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UNRAVELING THE DYNAMICS OF HYBRID ORDER DIMENSIONS

Athanasios Andrikopoulos[1]

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Abstract
Zeno's paradox, originating in antiquity, ignited the debate over whether time should be conceived as a discrete or a continuous entity. The notion of an instant devoid of duration represents a culmination of centuries of contemplation, spurred by Zeno's enigma. Time, conventionally, is understood as an entity isomorphic to real numbers, and in light of every experience possessing some duration, we tend to conceive of times as either devoid-of-duration instants or as assemblages of such instants. Consequently, the customary approach involves defining time intervals using instant time points and their relative precedence (sets of instant time points). A dissenting perspective was championed by Russell, who proposed an inverse formulation: temporal instants should be constructed from what he termed events. His aim was particularly focused on deducing an instant of time (or a point on a line) from a period of time (or from an interval on this line). Wiener, in his seminal work [28], introduces an axiomatic framework addressing Russell's conundrum, enabling the definition of instants. To achieve this, he establishes a precedence relation defined on a set of events, subject to a specific condition:

1 Introduction

Zeno's paradox posed for the first time the question of whether time should be represented by a discrete or a continuous variable. The concept of a durationless temporal instant is quite sophisticated, the result of many centuries of experimentation in response to Zeno's puzzle. Time has long been accepted as a structure isomorphic to real numbers and since any experience has some duration, we've come to think of times as either durationless instants or collections of such instants. As a result, it is standard practice to define time intervals using instant time points and their precedence relationship (sets of instant time points). Russell, however, proposed to go the other way around: temporal instants should be constructed from what he calls events. He wanted especially to derive, an instant of time (or of a point on a line) from a period of time (or from an interval on this line). In his paper [28], Wiener provides an axiomatic frame for Russell's problem in which instants can be defined. To do that, he defines a precedence relation R defined on a set of events X satisfying the following condition:

$\forall a,b,c,d \in X,\ (a,b) \in R,\ (c,b) \notin R$ and $(c,d) \in R$ imply $(a,d) \in R$ $(\star)$

where $(x,y) \in R$ means that $x,y \in X$ and x temporally wholly precedes y, i.e., every time at which x exists is temporally precedent to any time at which y exists. Russell and Wiener postulate that for each $x \in X, (x,x) \notin R$

[1] Dept. of Computer Engineering and Informatics, University of Patras, Patras, 26504, Greece

holds. We shall call statement $(\star)$ the *Russell - Wiener axiom*. Intuitively the formula states that if a precedes b and $b$is simultaneous with c, and c procedes$d$, then$a$ precedes d. The name interval order for these relations first appeared by Fishburn [6], [7]. Interval orders are important special classes of strict partial orders that arise in problems in graph theory, computer science, economics, psychology, biology, scheduling, and so on. For example, interval orders and the graph theory associated with their incomparability graphs, also called interval graphs, provide a natural model for the study of scheduling and preference models. Interval orders also have applications in distributed computing (vector clocks and global predicate detection), concurrency theory (pomsets and occurrence nets), programming language semantics (fixed-point semantics), data mining (concept analysis), etc.

Generally, for many applications in computer science, the precise time of each event occurrence is usually not needed, but what really counts is the precedence relation. In most of these cases, the precedence relation holds for events a b $a$and if ends before$b$ begins, and thus according to this logic, we can construct a time model where each event corresponds to an interval representing its duration. In this case, two events are incomparable if their temporal durations overlap. By using the Russell-Wiener axiom, the transitivity of the precedence's axiom and the notion of overlapping intervals allow us to infer information regarding the sequence of events. Let's see an example which illustrates the use of interval orders in computer science. In scheduling modeled by precedence constraints, we have several tasks, say, $t_1, t_2, \dots, t_n$ which have to be executed by a number of parallel processors $p_1, p_2, \dots, p_n$. We are assuming that all processors are identical, and all tasks are known in advance and can be executed independently from each other. Each assignment of tasks to processors is called a *schedule*. The sum of the processing times of the tasks, assigned to a processor, is the *load* of this processor and, the maximum load of any processor is the *length* of the schedule. Our strategy here is an optimal schedule, that is, a schedule of minimal length. In the case where the precedence constraint is an interval order, Papadimitriou and Yannakakis [16] showed that if tasks are put into a list sorted by non-increasing size of successorsets, and whenever a processor becomes idleit executes the leftmost unscheduled task in the list that is ready for execution, then one obtains a schedule of minimal length (see also [20, Page 3]). Finally, if an interval order R represents the time intervals for a given set of tasks, the breadth (the maximum size of an antichain in R) gives an upper bound on how many tasks are running at thesame time. This has applications, for example, in register allocation on a computer CPU.

On the other hand, it is commonly known that graphs are a powerful tool for modeling problems that arise in all areas of our life. A graph $G = (V, E)$ is called an intersection graph for a non-empty family $\mathcal{F}$ of geometric objects if there is a one-to-one correspondence between $\mathcal{F}$ $V$and such that two geometric objects in $\mathcal{F}$have non-empty intersection if and only if their corresponding vertices in$V$ are adjacent. Such a family of geometric objects is called an intersection representation of the graph. One of the most important intersection graphs are that of intervals on the real line and that of triangles defined by a point on a horizontal line and an interval or a unit interval on another horizontal line. Intersection graphs have natural applications in several fields, including bioinformatics and involving the physical mapping of DNA and the genome reconstruction.

A partially ordered set or poset, $(X, <)$, consists of a set X together with an irreflexive and transitive binary relation$<$ on it. A realizer of a poset$(X, <)$ is a family of linear orders on X whose intersection is the binary relation R. Szprilrajn [21] first proved that a realizer for a partial order R always exists. Dushnik and Miller [3] defined the order dimension $dim(\mathrm{R})$ of a poset$(X, R)$to be the minimum cardinality of a realizer. The concept of order dimension plays a role that in many instances is analogous to the chromatic number for graphs. Spinrad [19] believes that order dimension is a parameter that in some sense measures the complexity of a partial order. In fact, various problems may be easier to solve when restricted to partial orders of small order dimension. There are efficient algorithms for determining whether a partial order has at most two-order dimension. In 1982, Yannakakis [29] showed that testing if a partial order has order dimension $\leq k$, where $k \geq 3$, isNP-complete. Dimension seems to be a particularly hard NP-complete problem. This is indicated by the fact that we have no heuristics or approximation algorithms to produce realizers of partial orders that have reasonable size (for details see [4], [8], [12], [19], [20], [26], [29]). The interval order dimension and semiorder dimension of a poset (X, R), denoted $idim(\mathrm{R})$ and $sdim(\mathrm{R})$, are defined analogously to the order dimension but with interval orders and semiorders instead of linear orders. Since strict linear orders are semiorders and semiorders are interval orders, we trivially obtain that order dimension is an upper bound and interval dimension is a lower bound for semiorder dimension. The dimension of acyclic binaryrelations R which are the intersection of orders from the same class $\mathcal{P}$ has been extensively investigated. In contrast, not much is known about dimension of acyclic binary relations R which are the intersection of orders from different classes $\mathcal{P}_1$ and $\mathcal{P}_2$. Two main examples in this area are linear-interval orders (resp. linear-semiorders) R, i.e., acyclic binary relations where $R = R_1 \cap R_2$, with R_1 being a linear order and R_2 being an interval order (resp. semiorder). The linear-interval (resp. linear-semiorder) dimension is defined analogously to the order dimension but with linear-interval orders (resp. linear-semiorders) instead of linear orders (see [13], [17], [22] and [23]).

In this paper, we give three main results on: (i) the (linear-) interval order and (linear-) semiorder extensions of a binary relation; (ii) the existence of a realizer of a (linear-) interval order and (linear) semiorder of a binary relation; and (iii) the characterization of the (linear-) interval order and (linear-) semiorder dimension of a binary relation.

These results give an analogue of the: (i) Szpilran extension theory for posets[21], (ii) Dushnik and Miller [3] measure of poset complexity (order dimension) and (iii) Hiraguchi [10], Ore [15] and Milner and Pouzet [14] characterization of order dimension for posets, in the hybrid order case.

2 Notations and definitions

Let X be a non-empty universal set of alternatives and $R \subseteq X \times X$ be a binary relation on X. We sometimes abbreviate $(x, y) \in R$ as xRy. An abstract system [27] is a pair (X, R), where X is a set and R is a binary relation such that given $x, y \in X, xRy$ means that x dominates y. We say that R on X is (i) *reflexive* if for each $x \in X, (x, x) \in R$; (ii) *irreflexive* if we never have $(x, x) \in R$; (iii) asymmetric if for all $x, y \in X, (x, y) \in R \Rightarrow (y, x) \notin R$; (iv) *transitive* if for all $x, y, z \in X, [(x, z) \in R$ and $(z, y) \in R] \Rightarrow (x, y) \in R$; (v) *antisymmetric* if for each $x, y \in X, [(x, y) \in R$ and $(y, x) \in R] \Rightarrow x = y$; (vi) *total* if for each $x, y \in X, x \neq y$ we have xRy or yRx. Let $\mathcal{B}$ be the set of binary relations on X. The *diagonal relation* Δ on X is defined by $\Delta = \{(x, x) \mid x \in X\}$. A *unary operator* ρ is a mapping from $\mathcal{B}$ $\mathcal{B}$ to . Thus, given a binary relation $R, \rho(R) \in \mathcal{B}$ is a R binary relation. We first define the basic unary operator for binary relations. Given a binary $P(R)$ relation R, the *asymmetric part* $P(R)$ of is defined as follows:

$$= x, y \in \times \mid x, y \in \{(\quad) \quad X \quad X \; ( ) \quad R \quad (\quad) \notin R\} \text{ and } y, x \quad .$$

A *closure operator* is a unary operator φ from $\mathcal{B}$ to $\mathcal{B}$ that satisfies the following three properties: for all $R, R' \in \mathcal{B}$, (a) $R = \varphi(R)$ (*extensiveness*); b() $R \subseteq R' \Rightarrow \varphi(R) \subseteq \varphi(R)'$ (*monotonicity*) and (c) $\varphi(\varphi(R)) = \varphi(R)$ (*idempotence*). For a particular property $\mathcal{P}$, a closure operation of R is defined to be the smallest relation R_0 that contains R and has the desired property $\mathcal{P}$. Now, we provide two examples of closure operations. First, the *transitive closure* of a relation R is denoted by $\bar{R}$, that is, for all $x, y \in X, (x, y) \in \bar{R}$ if there exists $m \in \mathbb{N}$ and $z_0, \ldots, z_m \in X$ such that $x = z_0, (z_k, z_{k+1}) \in R$ for all $k \in \{0, \ldots, m-1\}$ and $z_m = y$. Clearly, $\bar{R}$ is transitive and because the case $m = 1$ is included, it follows that $R \subseteq \bar{R}$. Secondly, the *reflexive closure* of R is defined as follows:

$rc(R) = R \cup \Delta$.

The following combinations of properties are considered in the next theorems. A binary relation R on X is: (1) a *strict partial order* if R is irreflexive and transitive; (2) a *partial order* if R is reflexive, transitive and antisymmetric; (3) an *interval order* if R is a strict partial order which satisfies the Russell-Wiener axiom; (4) a *strong interval order* (see [2, Definition 3]) if R is the reflexive closure of an interval order $(R = rc(Q)$ where Q is an interval order); (5) a *strict linear order* if R is a total strict partial order and (6) a *linear order* if R is a total partial order. A subset $Y \subseteq X$ is an R-

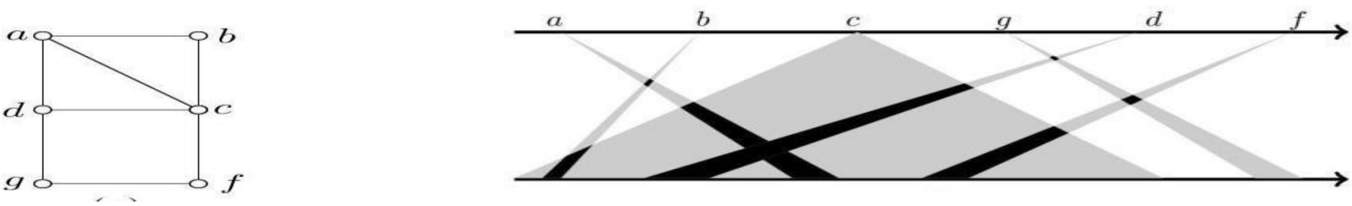
cycle if, for all $x, y \in Y$, we have $(x, y) \in \bar{R}$and $(y, x) \in \bar{R}$. We say that R is *acyclic* if there does not exist an$R$-cycle. A binary relation R^* is an *extension* of a binary relation R if and only if $R \subseteq R^*$and $P(R) \subseteq P(R^*)$. The (*interval) order dimension* of a partially ordered set $(X, \prec)$ is the leastλ such that there areλ (interval order) linear order extensions of whose intersection is$\prec$.

Since a linear order is a special case of an interval order and of a semiorder respectively, we conclude that a linear order extension of a binary relation R is also an interval order as well as a semiorder extension of R. The converse is not true. In the simple example which follows this can be confirmed.

Example 2.1. Let $\underset{\sim}{X} = \{x_1, x_2, x_3, x_4\}$ be a set and let $R_1 = \{(x_1, x_2), (x_3, x_4)\}$ and $R_2 = \{(x_1, x_2), (x_1, x_3), (x_2, x_3)\}$ be two relations onX. Then, $R_1 = \{(x_1, x_2), (x_3, x_4), (x_1, x_4)\}$is an interval order extension of$R$which is not a linear order and $\tilde{R}_2 = \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_4, x_3)\}$is a semiorder extension ofRwhich is not a linear order as well.

Cerioli, Oliveira and Szwarcfiter in [2] gave a common generalization of interval order dimension and (linear) order dimension of partial order $\precsim$. We extend this generalization in acyclic binary relations as follows: An acyclic binary relation R is called a *linear-interval order* if there exist a linear order L and an interval order Q such that

$\bar{R} = L \cap Q$. In this direction, we call an acyclic binary relation a *linear-semiorder* if its transitive closure is the intersection of a linear order and a semiorder (see [24]). Suppose$\mathcal{S} = \{S_i \mid i \in I\}$ be a family of geometric objects. A graph $G = (V, E)$ is an *intersection graph* if we can associate $\mathcal{S}$ to$G$such that each S_i corresponds to a vertex in V and $(x, y) \in E$ if and only if the S_i corresponding to x and y have non-empty intersection. That is, there is a one-to-one correspondence between $\mathcal{S}$and$G$such that two sets in$\mathcal{S}$ have non-empty intersection if and only if their corresponding vertices in G are adjacent. Intersection graphs are vital from both theoretical and practical points of view.An interval graph is the intersection graph of a family of intervals of the real line, called an interval model. Let L_1andL_2 be two distinct parallel lines. A *permutation graph* is the intersection graph of a family of line segments whose endpoints lie on two parallel lines L_1andL_2. A trapezoid graph is the intersection graph of a family of trapezoids $ABCD$, such that $AB$is on$L_1$and CDonL_2.A point-interval graph (orPIgraph) is the intersection graph of a family of trianglesABC, such that$A$ is onL_1andBC is on L_2.Figure 1 illustrates a*PI* graphwhere L_1 is represented by the top line and L_2 by the bottom line. Point-interval graphs generalize both permutation and interval graphs and lie between permutation and trapezoid graphs.



(a) (b)

Figure 1: (a) A simple-triangle graph of G. (b) An intersection representation of G.

In fact, an acyclic binary relation$R$ is called a *linear-interval order* if for each $x \in X$ there exists a triangle $T(x)$ such that $x\bar{R}y$ if and only if$T(x)$ lies completely to the left of$T(y)$.

In fact, the ordering of the apices of the triangles gives the linear order L, and the bases of the triangles give an interval representation of the interval order I. Let $\mathcal{K}$ be a family of geometric objects on X and let L_1andL_2 be two horizontal lines in thexy-plane with L_1above L_2. Generally speaking, a binary relation Ron a set X is $\mathcal{K}$*-order* if for each element $x \in X$, there is a geometric object $\mathcal{K}$ between L_1 and L_2 so that for any two elements $x, y \in X$, we have $x \prec y$in R if and only if$\mathcal{K}(x)$ lies completely to the left of $\mathcal{K}(y)$. The set $\{\mathcal{K}(x) \mid x \in X\}$is called a$\mathcal{K}$ representation of

R. Linear-interval orders have a triangle representation and Linear-semiorders have a unit triangle representation.

We say that $\mathcal{R}$ is a(p,q)-*linear-interval realizer*of R, if$\mathcal{R}$ is an interval realizer of $\bar{R}(\bar{R} = \cap\mathcal{R})$ with p elements and precisely q of them are non-linear. In this case we say that $\mathcal{R}$ (p,q)-realize R. We define $(p,q) \leq (p',q')$ if(p,q) is lexicographically smaller than or equal to (p',q'). A linear-*interval dimension* of an order$R$, denoted by$lidim(R)$, is the lexicographically smallest ordered pair (p,q) such that there exists a(p,q)-*linear-interval* realizer of R (see [2, Page 113]). Similarly, we define the notion (p,q)-linear-semiorder realizer of R.

3 Main result

Szpilrajn's extension theorem shows that any irreflexive and transitive binary relation has an irreflexive, transitive and total (strict linear order) extension (see Szpilrajn [21]). A general result of Szpilrajns extension theorem is the following corollary.

Corollary 3.1. A binary relation Ron a set X has a strict linear order extension if and only if R is an acyclic binary relation.

Proof. To prove the necessity of the corollary, we assume that R is acyclic. Then, $\bar{R}$ is irreflexive and transitive. By Szpilrajns extension theorem $\bar{R}$ has a strict linear order extension R^*. Since $R \subseteq \bar{R}$ we have that R^* is a strict linear order extension of R. To prove the sufficiency, let us assume that$R$ has a strict linear order extension Q^*.Then, R is acyclic. Indeed, suppose to the contrary that there exist $x,y \in X$such that $x\bar{R}y$and$y\bar{R}x$. It follows that xQ^*x, a contradiction to the irreflexivity of Q^*. The last conclusion completes the proof.

Szpilrajn's result remains true if asymmetry is replaced with reflexivity and antisymmetric (see [1, Page 64], [9]), that is, every reflexive, transitive and antisymmetric binary relation has a linear order extension. We generalize this result as follows:

Definition 3.1. A binary relation$R$ on a set X is *transitively antisymmetric* if and only if $\bar{R}$ is antisymmetric.

Proposition 3.2. A binary relation R on a set X has a linear order extension if and only if$R$istransitively antisymmetric.

Proof. To prove the necessity of the proposition, we assume that R is transitively antisymmetric. Then, $\bar{R}$ is transitive and antisymmetric. Then, by Arrow [1, Page 64] and Hansson [9],$\bar{R}$has a linear order extension. Therefore,R has a linear order extension. To prove the sufficiency, suppose that R has a linear order extension. If$R$is not transitively antisymmetric, then there are $x,y \in X$such that $(x,y) \in \bar{R}, (y,x) \in \bar{R}$and $x \neq y$. But then, $(x,y) \in Q, (y,x) \in Q$and$x \neq y$ which is impossible by the antisymmetry of Q.The last contradiction shows that$R$ is transitively antisymmetric.

To continue the study on the interval order dimension let us make the following assumption.

Negative interval order assumption. Let a binary relation Ron$X$ be given. Then, there exists $x,y,a,b \in X$ such that $(x,a) \in R, (b,y) \in R, (b,a) \notin \bar{R}and(x,y) \notin R$hold. The set

$\mathfrak{D}_R = \{((x,y),(a,b)) \in X^2 \times X^2 \mid (x,a) \in R, (b,y) \in R, (b,a) \notin \bar{R}$ and $(x,y) \notin R\}$

is called *the negative interval order assumption set with respect to*R.

Negative semiorder assumption. Let a binary relation Ron$X$ be given. Then, there exists $x,y,z,w \in X$ such that $(x,y) \in$
$R, (y,z) \in R, (x,w) \notin R$ and $(w,y) \notin R$ hold.

Remark 3.3. If a binary relation$R$ is assumed to satisfy the negative interval order assumption generalizes the $2+2$rule and if it is assumed to satisfy the semiorder assumption is equivalent to fulfil the $3+1$ rule. In this paper, we
use the first notation which is more convenient for presentation of proofs.

Lemma 3.4. Let$R$be an acyclic binary relation on a set$X$, which does not satisfy the negative interval order assumption. Then,$\bar{R}$is an interval order extension of$R$(not necessarily strict linear order).

Proof. By definition,$R \subseteq \bar{R}$ $\bar{R}$ and is transitive. Since $R$is acyclic, we also seethat$\bar{R}$ is irreflexive. To complete the proof, weonly need to verify that$\bar{R}$satisfies the Russell-Wiener axiom. In fact, since$R$does not satisfy thassumption of the negative interval order, we are led to conclude that for all$x,y,a,b \in X$, which satisfy

xRa, bRy and $(b,a) \notin \bar{R}$, we have $(x,y) \in R$. Let now $z,w,c,d \in X$ such that $z\bar{R}c, d\bar{R}w$ and $(d,c) \notin \bar{R}$. Then, there exist natural numbers μ, ν and alternatives $s_1, s_2, \ldots, s_\mu, t_1, t_2, \ldots, t_\nu$ such that

$zRs_1Rs_2 \ldots Rs_\mu Rc$ *and* $dRt_1Rt_2 \ldots Rt_\nu Rw$.

But then, $s_\mu Rc, dRt_1$ and $(d,c) \notin \bar{R}$ imply that $(s_\mu, t_1) \in R$. It follows that $(z,w) \in \bar{R}$. Hence, $\bar{R}$ is an interval order extension of R.

Theorem 3.5. A binary relation R on a set X has an interval order extension (not necessarily a strict linear order) if and only if R is acyclic.

Proof. Let us prove the necessity of the theorem. We assume that R is an acyclic binary relation defined in a set X. If R is an interval order (if $x,y,a,b \in X$ such that $xRa, bRy, (b,a) \notin R = \bar{R}$, then $(x,y) \in R$), then there is nothing to prove. Otherwise, $\mathfrak{D}_R \neq \emptyset$. That is, there exists $x,y,a,b \in X$ such that $xRa, bRy, (b,a) \notin \bar{R}$ and $(x,y) \notin R$. We put $R' = R \cup \{(x,y) \in X \times X \mid \exists a,b \in X$ such that xRa, bRy and $(b,a) \notin \bar{R}\}$.

Clearly, R' is irreflexive and $R \subset R'$. To verify that R' is acyclic, take any $z \in X$ and suppose that $(z,z) \in \bar{R}$. Then, there exists a natural number m and alternatives $x_1, x_2, \ldots, x_m$ such that

$z = x_1 R' x_2 \ldots R' x_{m-1} R' x_m = z$.

Since R is acyclic, there is at least one $k \in \{1, \ldots, m-1\}$ such that $(x_k, x_{k+1}) = (x,y)$ with $(x,y) \in R' \setminus R$. Let x_{k^*} be the first occurrence of x and let x_{l^*} be the last occurrence of y. Clearly, for all $k \in \{1, \ldots, m-1\}$, if $(x_k, x_{k+1}) \neq (x,y)$, then $(x_k, x_{k+1}) \in R$. Then,

$y = x_{l^*} R x_{l^*+1} \ldots RzRx_1 \ldots Rx_{k^*} = x$.

It follows that $(y,x) \in \bar{R}$ which jointly with $(x,a) \in R$ and $(b,y) \in R$ implies that $(b,a) \in \bar{R}$, causing an absurdity. Therefore, R' is acyclic. On the other hand, if R' does not satisfies the negative interval order assumption, then Lemma 3.4 implies that $\bar{R}$ is an interval order extension of R, which ends the proof of the necessity of the theorem. Otherwise, we proceed by assuming that R' satisfies the negative interval order assumption. Now, let us $\mathcal{E} = \{Q \subseteq X \times X \mid Q$ is an acyclic extension of R which satisfies the negative interval order assumption$\}$.

We have $R' \in \mathcal{E}$, so this class is not nonempty. Let $\mathcal{C} = (Q_\theta)_{\theta \in \Theta}$ be a chain in $\mathcal{E}$ and let $\hat{Q} = \cup_{\theta \in \Theta} Q_\theta$. Then, $\hat{Q} \in \mathcal{E}$.

To prove it, we first show that $\hat{Q}$ is acyclic (resp. irreflexive). Take $(x,x) \in \bar{\hat{Q}}$ (resp. $(x,x) \in \hat{Q}$) for some $x \in X$. Then, since $\mathcal{C}$ is a chain, there exists an $Q_{\theta^*} \in \mathcal{C}, \theta^* \in \Theta$ such that $(x,x) \in \overline{Q_{\theta^*}}$ (resp. $(x,x) \in Q_{\theta^*}$). This is impossible due to the acyclicity (irreflexivity) of Q_{θ^*}. Therefore, Q is irreflexive and acyclic. On the other hand, we assume that $\hat{Q}$ satisfies the negative interval order assumption, because otherwise Lemma 3.4 implies that $\bar{\hat{Q}}$ is an extension of the interval order of R, which ends the proof of the necessity of the theorem. Since $R \subset \hat{Q}$ we have that $\hat{Q} \in \mathcal{E}$. Therefore, any chain in $\mathcal{E}$ has an upper bound in $\mathcal{E}$ (with respect to set inclusion). By Zorn's lemma, there is a maximal element Q^* in $\mathcal{E}$. We prove that $\overline{Q^*}$ is an interval order extension of R. Clearly, $\overline{Q^*}$ is an irreflexive and transitive extension of R. It remains to prove that $\overline{Q^*}$ satisfies the Russell-Wiener axiom.

We proceed by way of contradiction. Suppose there are $x,y,a,b \in X$ such that $(x,a) \in \overline{Q^*}, (b,y) \in \overline{Q^*}, (b,a) \notin \overline{Q^*}$ and $(x,y) \notin \overline{Q^*}$. Then, $\overline{Q^*} \supset Q^*$ is an acyclic extension of R which satisfies the negative interval order assumption, a contradiction to the maximal character of Q^*. Clearly, in any case of proof, the extension of the interval R is not required to be of linear order. Thus, the last contradiction completes the necessity of the theorem.

To prove the sufficiency, let us assume that R has a not necessarily linear interval order extension Q^*. Then, R is acyclic. Indeed, suppose to the contrary that there exist $x \in X$, a natural number m and alternatives $x_1, x_2, \ldots, x_m$ such that

$xRx_1Rx_2 \ldots Rx_m Rx$.

Since Q^* is transitive and $R \subseteq Q^*$, we have xQ^*x, a contradiction to irreflexivity of Q^*. The last conclusion completes the proof.

Corollary 3.6. A binary relation R on a set X has a strong interval order extension ((not necessarily a linear order) if and only if R is transitively antisymmetric.

Proof. To prove the necessity of the corollary, we assume that R is transitively antisymmetric. Then, $\bar{R} \setminus \Delta$ is acyclic. By Theorem 3.5, $R \subseteq \bar{R}$ $(\bar{R} \setminus \Delta) \cup (\bar{R} \cap \Delta)$ $\bar{R} \setminus \Delta$ has an interval order extension R^*. Then, we have $= \subseteq R^* \cup (\bar{R} \cap \Delta) \subseteq R^* \cup \Delta$.

Therefore, $Q = rc(R^*) = R^* \cup \Delta$ is a strong interval order extension of R. To prove the sufficiency, let us assume that R has a strong interval order extension Q. Suppose on the contrary, that there are $x, y \in X$ such that $(x, y) \in \bar{R}$, $(y, x) \in \bar{R}$ and $x \neq y$. It follows that $(x, y) \in Q, (y, x) \in Q$ and $x \neq y$ which is impossible by the asymmetry of $Q \setminus \Delta$. The last contradiction completes the proof.

Theorem 3.7. A binary relation R in a set X is a linear interval order if and only if R is acyclic.

Proof. To prove the necessity of the theorem, let us suppose that R is an acyclic binary relation defined on a set X.

By Theorem 3.5 there exists an interval order extension Q of R (Q is not necessarily a strict linear order). Then, $\bar{R} \subseteq Q$ which implies that Q is an interval order extension of $\bar{R}$. We put

$R^* = \bar{R} \cup \{(x, y) \in X \times X \mid (y, x) \in Q \setminus \bar{R}\}$.

Since R is acyclic and Q is irreflexive, we have R^* is irreflexive. If $Q = \bar{R}$, then $R^* = \bar{R}$. By the Szpilrajn theorem, $\bar{R}$ has a strict linear-order extension L. It follows that $\bar{R} = Q \cap L$ which implies that R is a linear-interval order. Now suppose $Q \setminus \bar{R} \neq \emptyset$. We now prove that R^* is acyclic and thus is an acyclic extension of $\bar{R}$. Indeed, suppose to the contrary that there are alternatives $v, z_0, z_1, z_2, \dots, z_m \in X$ such that

$v = z_0 \; z_1 \; z_2 \dots {}^* z_m = v R^* \quad R^* \quad R$.

Since R is acyclic, there is at least one $\kappa \in \{0, 1, \dots, m-1\}$ such that $(z_\kappa, z_{\kappa+1}) = (x, y)$. Let z_{κ^*} be the first occurrence of x and let z_{λ^*} be the last occurrence of y. Then,

$y = z_{\lambda^*} \; z_{\lambda^*+1} \dots \; z_m = v = z_0 \; z_1 \; \dots \; z_{\kappa^*} = x R \quad R \quad R \; R \; R$.

It follows that $(y, x) \in \bar{R}$, a contradiction to $(y, x) \in Q \setminus \bar{R}$.

Suppose that $\tilde{\mathcal{R}} = \{\tilde{R}_i \mid i \in I\}$ denotes the set of acyclic extensions of $\bar{R}$ such that $(x, y) \in \tilde{R}_i \setminus \bar{R}$ if and only if $(y, x) \in Q \setminus \bar{R}$. Since $R^* \in \tilde{\mathcal{R}}$ we have that $\tilde{\mathcal{R}} \neq \emptyset$. Let $\mathcal{C} = (C_i)_{i \in I}$ be a chain in $\tilde{\mathcal{R}}$, and let $\hat{C} = \cup_{i \in I} C_i$. We prove that $C \in \mathcal{R}$. To prove that $\hat{C}$ is acyclic suppose to the contrary that there exists $\mu, s_0, s_1, s_2, \dots, s_n \in X$ such that

$\mu = s_0 \hat{C} s_1 \hat{C} s_2 \dots \hat{C} s_n = \mu$.

Since $\mathcal{C}$ is a chain, there exists $i^* \in I$ such that

$\mu = s_0 C_{i^*} s_1 C_{i^*} s_2 \dots C_{i^*} s_n = \mu$,

contradicting the acyclicity of C_{i^*}. On the other hand, it is easy to check that $(x, y) \in \hat{C} \setminus \bar{R}$ implies $(y, x) \in Q \setminus \bar{R}$.

By Zorn's lemma $\tilde{\mathcal{R}}$ possesses an element, say $\hat{R}$, that is maximal with respect to set inclusion. We have two cases to consider: $\hat{R}$ is total or not. If $\hat{R}$ is total, then $\hat{R}$ is a strict linear order extension of $\bar{R}$. Then, $\bar{R} = Q \cap \hat{R}$. Indeed, since $\bar{R} \subseteq Q \cap \hat{R}$, one needs only to prove that $Q \cap \hat{R} \subseteq \bar{R}$. Let to the contrary be $(x, y) \in Q \cap \hat{R}$ and $(x, y) \notin \bar{R}$. The $(x, y) \in Q \setminus \bar{R}$ which implies that $(y, x) \in \hat{R}$, a contrsdiction to asymmetry of $\hat{R}$ (irreflexive and transitive). Therefore,

$\bar{R} = Q \cap \hat{R}$. If $\hat{R}$ is not total, then there exists $x, y \in X$ such that $(x, y) \notin \hat{R}$ and $(y, x) \notin \hat{R}$. It follows that $(x, y) \notin \bar{R}$ and $(y, x) \notin \bar{R}$. But then, $(x, y) \notin Q$ and $(y, x) \notin Q$, because otherwise $(x, y) \notin Q \setminus \bar{R}$ or $(y, x) \notin Q \setminus \bar{R}$ which implies that $(y, x) \notin \hat{R}$ or $(x, y) \notin \hat{R}$ which is impossible. Since $\hat{R}$ and transitive, by the Szpilrajn theorem there exists a strict linear order extension $\hat{\hat{R}}$ of R. Since $(\hat{\hat{R}} \setminus \hat{R}) \cap Q = \emptyset$ we conclude that $Q \cap \hat{\hat{R}} = \bar{R}$. The last conclusion shows that R is a linear-interval binary relation. The converse is similar to the proof of the converse of Theorem 3.5.

Theorem 3.8. A binary relation R on a set X has a semiorder extension if and only if R is acyclic. *Proof.* Let R be an acyclic binary relation on X. By Theorem 3.5 has an interval order extension Q of R. Put

$Q^* = Q \cup \{(x, w) \in X \times X \setminus \Delta \mid$ there exist $y, z \in X$ such that $(x, y) \in Q, (y, z) \in Q, (x, w) \notin Q$ and $(w, z) \notin Q\} = Q \cup T$.

Clearly, Q^* is irreflexive. We prove that Q^* is transitive. Indeed, let $a, b, c \in X$ such that $(a, b) \in Q^*$ and $(c, d) \in Q^*$.

Then, we have four cases to consider:

Case 1_α. $(a,b) \in Q$ and $(b,c) \in Q$. Then, $(a,c) \in Q \subseteq Q^*$.

Case 2_α. $(a,b) \in Q$ and $(b,c) \in T$. Therefore, $(a,b) \in Q$ and there exists $\kappa, m, \lambda \in X$ such that $(b,\kappa) \in Q, (\kappa,\lambda) \in Q, (b,c) \notin Q$ and $(c,\lambda) \notin Q$. From $(a,b) \in Q$ and $(b,\kappa) \in Q$ we have that $(a,\kappa) \in Q$. If $(a,c) \in Q \subseteq Q^*$ we have nothing toprove. We suppose that $(a,c) \notin Q$. Then, from $(a,\kappa) \in Q$, $(\kappa,\lambda) \in Q, (a,c) \notin Q$ and $(c,\lambda) \notin Q$ we conclude $(a,c) \in Q^*$.

Case 3_α. $(a,b) \in T$ and $(b,c) \in Q$. In this case, we have $(b,c) \in Q$ and there exists $\kappa, \lambda \in X$ such that $(a,\kappa) \in Q, (\kappa,\lambda) \in Q, (a,b) \notin Q$ and $(b,\lambda) \notin Q$. Since $(\kappa,\lambda) \in Q, (b,c) \in Q$ and $(b,\lambda) \notin Q$ we conclude that $(\kappa,c) \in Q$ which jointly to $(a,\kappa) \in Q$ implies that $(a,c) \in Q \subseteq Q^*$.

Case 4_α. $(a,b) \in T$ and $(b,c) \in T$. In this case, there are $\kappa, \lambda, \nu, \mu \in X$ such that $(a,\kappa) \in Q, (\kappa,\lambda) \in Q, (a,b) \notin Q, (b,\lambda) \notin Q$ and $(b,\mu) \in Q, (\mu,\nu) \in Q$, $(b,c) \notin Q$ and $(c,\nu) \notin Q$. If $(a,c) \in Q \subseteq Q^*$, then we have nothing to prove. Suppose that $(a,c) \notin Q$. If $(c,\lambda) \notin Q$, then from $(a,\kappa) \in Q, (\kappa,\lambda) \in Q$ and $(a,c) \notin Q$ we conclude that $(a,c) \in T \subseteq Q$. Otherwise, if $(c,\lambda) \in Q$, then we have two subcases to consider when $(a,\mu) \in Q$ or not. If $(a,\mu) \in Q$, then from $(m,\nu) \in Q, (a,c) \notin Q$ and $(c,\nu) \notin Q$ we have $(a,c) \in T \subseteq Q^*$. On the other hand, if $(a,\mu) \notin Q$, then $(b,\mu) \in Q, (a,\kappa) \in Q$ implies that $(b,\kappa) \in Q$ which jointly to $(\kappa,\lambda) \in Q$ implies that $(b,\lambda) \in Q$ which is impossible. Therefore, in all possible cases $(a,c) \in Q^*$ which implies that Q^* is transitive.

To prove that Q^* is an interval order we have four cases to consider.

Case 1_β. $(a,b) \in Q, (c,d) \in Q$ and $(c,b) \notin Q^* \supseteq Q$. Since Q is an interval order, in this case It is clear that $(a,d) \in Q \subseteq Q^*$.

Case 2_β $(a,b) \in Q, (c,d) \in T$ and $(c,b) \notin Q^* \supseteq Q$. In this case, there are $\kappa, \lambda \in X$ such that $(c,\kappa) \in Q, (\kappa,\lambda) \in Q, (c,d) \notin Q$ and $(d,\lambda) \notin Q$. Then, from $(a,b) \in Q, (c,k) \in Q$ and $(c,b) \notin Q$ we conclude that $(a,k) \in Q$. If $(a,d) \in Q \subseteq Q^*$, then we have nothing to prove. If $(a,d) \notin Q$, then from $(a,\kappa) \in Q, (\kappa,\lambda) \in Q, (a,d) \notin Q$ and $(d,\lambda) \notin Q$ we have that $(a,d) \in T \subseteq Q^*$.

Case 3_β. $(a,b) \in T, (c,d) \in Q$ and $(c,b) \notin Q^* \supseteq Q$. In this case, we have $(c,d) \in Q$ and there exists $\kappa, \lambda \in X$ such that $(a,\kappa) \in Q, (\kappa,\lambda) \in Q, (a,b) \notin Q$, $(b,\lambda) \notin Q$ and $(c,b) \notin Q^* \supseteq Q$. If $(a,d) \in Q \subseteq Q^*$, then we have nothing to prove. Let $(a,d) \notin Q$. If $(d,\lambda) \notin Q$, then $(a,\kappa) \in Q, (\kappa,\lambda) \in Q$ implies $(a,d) \in T \subseteq Q^*$. Otherwise, $(d,\lambda) \in Q$ which jointly to $(c,d) \in Q, (c,b) \notin Q, (b,\lambda) \notin Q$ implythat $(c,b) \in T \subseteq Q^*$, a contradiction. Therefore, $(a,d) \in T \subseteq Q^*$.

Case 4_β $(a,b) \in T$ and $(b,c) \in T$. In this case, there are $\kappa, \lambda, \nu, \mu \in X$ such that $(a,\kappa) \in Q, (\kappa,\lambda) \in Q, (a,b) \notin Q, (b,\lambda) \notin Q, (c,\mu) \in Q, (\mu,\nu) \in Q$, $(c,d) \notin Q$ and $(c,\nu) \notin Q$ and $(c,b) \notin Q^* \supseteq Q$. If $(a,d) \in Q \subseteq Q^*$, then we have nothing to prove. Let $(a,d) \notin Q$. If $(d,\lambda) \notin Q$, then from $(a,\kappa) \in Q$, $(\kappa,\lambda) \in Q$ and $(a,d) \notin Q$ we conclude $(a,d) \in T \subseteq Q^*$. If $(d,\lambda) \in Q$, then we have two subcases to consider: (4_a) $(a,\mu) \in Q$ and (4_b) $(a,\mu) \notin Q$. If $(a,\mu) \in Q$, then from $(\mu,\nu) \in Q, (a,d) \notin Q$ and $(d,\lambda) \notin Q$ we conclude that $(a,d) \in T \subseteq Q^*$. If $(a,\mu) \notin Q$, then from $(c,m) \in Q$ and $(a,\kappa) \in Q$ we conclude that $(c,k) \in Q$. But then, $(c,k) \in Q, (\kappa,\lambda) \in Q, (c,b) \notin Q$ and $(b,\lambda) \notin Q$ implies that $(c,b) \notin Q^*$, anabsurdity. Hence, $(a,d) \in Q^*$. Therefore, Q^* is an interval order. If Q^* does not satisy the negative semiorder assumption, then Q^* is a semiorderextension of R and the proof is over. Otherwise, Q^* satisfies the negative semiorder assumption. Now, let $\mathcal{E} = \{Q \subseteq X \times X \mid Q$ is an interval order extension of R which satisfies the negative semiorder assumption$\}$.

We have $Q^* \in \mathcal{E}$, so this class is nonempty. Let $\mathfrak{C} = \left\{\left((C_\eta^i)_{\eta \in H_i}\right) \mid i \in I\right\}$ be the family of chains in $\mathcal{E}$. If $C_\eta^{i^*} = (Q_j)_{j \in J}$ is a chain in $\mathcal{E}$ such that $\hat{Q} = \cup_{j \in J} Q_j$ does not satisfy the negative semiorder assumption, then $\hat{Q}$ is a semiorder extension of R. Otherwise, for each $i \in I, \cup_{\eta \in H_i} C_\eta^i \in \mathcal{E}$ holds. By Zorn's lemma, there is a maximal element $\hat{Q}^*$ in $\mathcal{E}$. We prove that $\hat{Q}^*$ is a semiorder extension of R. Indeed, suppose to the contrary that $\hat{Q}^*$ is not a semiorder. Then, thereexist $x, y, w, z \in X$ such that $(x,y) \in \hat{Q}^*, (y,z) \in \hat{Q}^*, (x,w) \notin \hat{Q}^*$ and $(w,z) \notin \hat{Q}^*$. But then, the relation $\hat{Q}^* = \hat{Q}^* \cup \{(x,w) \in X \times X \setminus \Delta \mid$ there exist $y, z \in X$ such that $(x,y) \in \hat{Q}^*, (y,z) \in \hat{Q}^*, (x,w) \notin \hat{Q}^* (w,z) \notin \hat{Q}\}$ and belongs to $\mathcal{E}$, a contradiction to the maximal character of $\hat{Q}^*$. Therefore, Q^* is a semiorder extension of R. The converse is evident.

The following theorem is proved in a similar way to the proof of Theorem 3.7.

Theorem 3.9. A binary relation R on a set X is a linear-semiorder if and only if R is acyclic.

4 Hybrid order dimension

Today, dimension theory is a strong advancement in graph theory and computer science. This is documented in the recent book by Trotter [25], which provides a comprehensive survey.

The notion of dimension of a poset $(X,\prec)$ was introduced in a seminal paper by Dushnik and Miller [3] as the least λ such that there are λ linear extensions of $\prec$ whose intersection is $\prec$. Equivalently, the dimension of $\prec$ is the dimension of the Euclidean space $\mathbb{R}^{\lambda}$ in which $(X,\prec)$ can be embedded in such a way that $x \prec y$ if and only if the point of x is below the point of y with respect to component wise order (see Ore [15]). In a more general context, we often have a class $\mathcal{R}$ of objects, e.g., acyclic binary relations, graphs, digraphs, specific kinds of them, etc.- and a subclass $\mathcal{C}$ $\mathcal{R}$ of such that every $R \in \mathcal{R}$ is either equivalent to the intersection of a number of $C_i \in \mathcal{C}$ or can be embedded into a product $\prod_{i<\lambda} C_i$ with $C_i \in \mathcal{C}$ and λ being a cardinal number. Then it is natural to regard the necessary number of C_λ as a measure of complexity of R, called the dimension of R with respect to $\mathcal{C}$ and $\mathcal{R}$. The following theorem is a generalized result to that of Dushnik and Miller, and it is a key result for the study of the interval order dimension.

Theorem 4.1. Let (X,R) be an abstract system. Then, $\bar{R}$ has as a realizer the set of interval order extensions of R if and only if R is acyclic.

Proof. To prove the necessity, let R be an acyclic binary relation on X and let $\mathcal{Q}$ be the set of all interval order extensions of R. By Theorem 3.5, the family of such extensions is non-empty. We show that $\bar{R} = \bigcap_{Q\in\mathcal{Q}} Q$. Clearly, $\bar{R} \subseteq \bigcap_{Q\in\mathcal{Q}} Q$. Therefore, we have only to show that $\bigcap_{Q\in\mathcal{Q}} Q \subseteq \bar{R}$. Suppose to the contrary that there exists a pair $(a,b) \in \bigcap_{Q\in\mathcal{Q}} Q$ but $(a,b) \notin \bar{R}$. We first show that $(b,a) \notin \bar{R}$. Indeed, if we suppose, for the sake of contradiction, that $(b,a) \in \bar{R}$, then we have $(b,a) \in \bar{Q} = Q$. This contradicts the fact that Q is asymmetric (irreflexive and transitive). Therefore, $a,b \in X$ are non-comparable with respect to $\bar{R}$. Put

$R' = \bar{R} \cup \{(b,a)\}$

It is easy to check that R' is acyclic $((a,b) \notin \bar{R})$. By Theorem 3.5, R' has an interval order extension Q^*. Therefore, R has an interval order extension Q^* such that $(b,a) \in Q^*$, a contradiction to the asymmetry of $Q^*((a,b) \in \bigcap_{Q\in\mathcal{Q}} Q \subseteq Q^*)$. The last contradiction proves that $\bar{R} = \bigcap_{Q\in\mathcal{Q}} Q$.

To prove the sufficiency of the theorem, let $\bar{R} = \bigcap_{Q\in\mathcal{Q}} Q$, where $\mathcal{Q}$ is a family of interval order extensions of R. Then, R is acyclic. Indeed, suppose to the contrary that there are alternatives $x, x_0, x_1, \ldots, x_n \in X$ such that

$x = x_0 R x_1 R \ldots R x_n = x$.

Since Q is a transitive extension of R, we have xQx, a contradiction to irreflexivity of Q. Therefore, R is acyclic. The last conclusion completes the proof.

The following corollary is a consequence of Theorem 4.1.

Corollary 4.2. Let (X,R) be an abstract system. Then, $\bar{R}$ has as realizer the set of strong interval order extensions of R if and only if R is reflexive and transitively antisymmetric.

Proof. To prove the necessity, let R be reflexive and transitively antisymmetric. Then, $\bar{R}\setminus\Delta$ is acyclic. By Theorem 4.1, we have that $\bar{R}\setminus\Delta = \bigcap_{Q\in\mathcal{Q}} Q$, where Q is an interval order. Therefore, $\bar{R} = \bigcap_{Q\in\mathcal{Q}} rc(Q)$ where $rc(Q)$ is a strong interval order. Conversely, suppose that $\bar{R}$ has as realizer the set $\mathcal{Q}^*$ of strong interval order extensions of R. If $Q^* \in \mathcal{Q}^*$, then $Q^*\setminus\Delta$ is an interval order. If we suppose that R is not transitively antisymmetric, then we conclude that $Q^*\setminus\Delta$ is not asymmetric, which is a contradiction. Therefore, R is transitively antisymmetric. On the other hand, since $\Delta \subseteq \bigcap_{Q^*\in\mathcal{Q}^*} Q^* = \bar{R}$, we see that for all $x \in X$ there is $(x,x) \in \bar{R}$. Thus, here are alternatives $x, x_0, x_1, \ldots, x_n \in X$ such that

$x = x_0 R x_1 R \ldots R x_n = x$.

Since R is transitively antisymmetric, we conclude that $x = x_0 = x_1 = \cdots = x_n$ which implies that $(x,x) \in R$. Hence, R is reflexive.

Moreover, if$R$ is transitive, then as immediate consequences of Theorem 4.1 and Corollary 4.2 we have the following results.

Corollary 4.3. A binary relation R has as realizer the set of its interval order extensions if and only if R is a strict partial order.

Corollary 4.4. A binary relation R has as realizer the set of its strong interval order extensions if and only if R is a partial order.

The following result is a generalization of the theorem of Dushnik and Miller [3].

Theorem 4.5. Let (X,R)be an abstract system. Then, $\bar{R}$ has as realizer the set of strict linear order extensions of R if and only if$R$ is acyclic.

Proof. Let R be an acyclic binary relation on X. Then, $(X,\bar{R})$ is a poset. By ([3, Theorem 2.32] we have that the family $\mathcal{Q}$ of strict linear order extensions of $\bar{R}$ is a realizer of$\bar{R}$. That is, $\bar{R} = \bigcap_{Q\in\mathcal{Q}} Q$. Since $R \subseteq \bar{R}$and $R \subseteq Q$imply $\bar{R} \subseteq \bar{Q} = Q$, we have that the family of strict linear order extension of $\bar{R}$coincides with the family of strict linear order extension of R.

Conversely, suppose that $\bar{R}$ has as realizer the set of strict linear order extensions of $R,\mathcal{Q}$. Then, $\bar{R} = \bigcap_{Q\in\mathcal{Q}} Q$.Since $\bigcap_{Q\in\mathcal{Q}} Q$ is irreflexive, we conclude that $\bar{R}$ is acyclic.

By analogy to the proof of Corollary 4.2 of Theorem 4.1, we can prove the following corollary from Theorem 4.5. **Corollary 4.6.** Let (X,R)be an abstract system. Then, $\bar{R}$ has as realizer the set of linear order extensions of R if and only if R is reflexive and transitively antisymmetric.

Moreover, if R is transitive, then as immediate consequences of Theorem 4.5 and Corollary 4.6 we have the following results.

Corollary 4.7. Let (X,R) be an abstract system. Then, R has as realizer the set of strict linear order extensions of R if and only if R is transitive and asymmetric.

Corollary 4.8. Let (X,R) be an abstract system. Then, R has as realizer the set of linear order extensions of$R$ if and only if R is reflexive, thansitive, and antisymmetric.

The following two theorems are proved in a similarway to the proof of Theorem 4.1.

Theorem 4.9. Let (X,R) be an abstract system. Then,$\bar{R}$ has as realizer the set of linear-interval order extensions of R if and only if R is acyclic.

Theorem 4.10. Let (X,R) be an abstract system. Then, $\bar{R}$ has as realizer the set of linear-semiorder extensions of Rif and only if R is acyclic.

As we mentioned above, Ore [15] defined order dimension of a poset$\mathcal{P} = (X,\prec)$ as the least cardinalλ (see also Hiraguchi [10]) such that there is an order preserving embedding of $(X,\prec)$ into a direct product $dpc(\mathcal{P}) = \otimes \{(X,\leq) \mid i<\lambda\}(\prod_{i<\lambda} X^i, <_Q)of\lambda$ linear orders $\leq_i$ $(i<\lambda)$,where $<_Q$ is defined by $(x_i)_{i<\lambda} \leq_Q (y_i)_{i<\lambda}$ if and only if $x_i \leq_i y_i$ holds for all $i<\lambda$.

On the other hand, Milner and Pouzet [14] proved that the dimension of a poset$\mathcal{P}$ is equal to the least cardinal λ such that there is an order preserving embedding of $(X,\prec)$ into a strict direct product $spc(\mathcal{P}) = \odot \{(X,<_i) \mid i<\lambda\} = (\prod_{i<\lambda} X^i, <_S)of\lambda$ strict linear orders $<_i$ $(i<\lambda)$, where $<_S$ is defined by $(x_i)_{i<\lambda} <_S (y_i)_{i<\lambda}$ if and only if$x_i <_i y_i$ holds for all$i<\lambda$.

In order to give general results concerning those of (interval) order dimension, we extend the notions of order preserving embedding, componentwise order and (strict) direct product of a partial order to arbitrary binary relations.

In the following, for the sake of maintaining uniformity of notations, for any abstract system (X,R)we denote $<_R = P(R)$and $\leq_R = P(R) \cup \Delta = rc(P(R))$. Clearly, if R is acyclic, then $<_R = R$and $\leq_R = rc(R)$.

Definition 4.1. A mapping from an abstract system (X,R) to an abstract system (X',R') is called an *dominancepreserving embedding* if it respects the dominance relation, that is, all $x,y \in X$ are mapped to $x',y' \in R'$ such that xRy if and only if $x'R'y'$. Let $\lambda \in \aleph$ be a cardinal number and let $\mathfrak{R} = \{(X_i,R_i) \mid i<\lambda\}$be a family of abstract

systems. The *strict componentwise dominance relation* of $\mathfrak{R}$ is a binary relation $S(\mathfrak{R})$ on the Cartesian product $\prod_{i<\lambda} X^i$ such that given $(x_i)_{i<\lambda}, (y_i)_{i<\lambda} \in \prod_{i<\lambda} X^i$, we have $(x_i)_{i<\lambda} <_{S(\mathfrak{R})} (y_i)_{i<\lambda}$ if and only if $x_i <_{R_i} y_i$ for all $i < \lambda$.
The *componentwise dominance relation* of $\mathfrak{R}$ is a binary relation $Q(\mathfrak{R})$ on the cartesian product $\prod_{i<\lambda} X^i$ such that given $(x_i)_{i<\lambda}, (y_i)_{i<\lambda} \in \prod_{i<\lambda} X^i$, we have
$(x_i)_{i<\lambda} \leq_{Q(\mathfrak{R})} (y_i)_{i<\lambda}$ if and only if $x_i \leq_{R_i} y_i$ for each $i < \lambda$.
The *strict direct product* of a family $\mathfrak{R} = \{(X_i, R_i) \mid i < \lambda\}$ of abstract systems, denoted by $\odot \{(X, \mathcal{R}_i) \mid i < \lambda\}$, is the Cartesian product $\prod_{i<\lambda} X^i$ equipped with the strict componentwise dominance relation $<_{S(\mathfrak{R})}$. In this case, we write $(\tilde{X}, <_{S(\mathfrak{R})}) = \odot \{(X, \mathcal{R}_i) \mid i < \lambda\}$ where $\tilde{X} = \prod_{i<\lambda} X^i$. The *direct product* of a family $\mathfrak{R} = \{(X_i, R_i) \mid i < \lambda\}$ of abstract systems, denoted by $\otimes \{(X, \mathcal{R}_i) \mid i < \lambda\}$, is the Cartesian product $\prod_{i<\lambda} X^i$ equipped with the componentwise dominance relation $<_{Q(\mathfrak{R})}$. In this case, we write $(\tilde{X}, <_{Q(\mathfrak{R})}) = \otimes \{(X, \mathcal{R}_i) \mid i < \lambda\}$ where $\tilde{X} = \prod_{i<\lambda} X^i$.
In case of (strict) partial orders, the notions of dominance-preserving embedding, componentwise dominance relation and (strict) direct product of an abstract system coincide with the notions of order-preserving embedding, componentwise order and (strict) direct product of linearly ordered sets, respectively.
We now extend the notion of order dimension to study the problem of (interval) order dimension in a general form.

Definition 4.2. Let $\mathfrak{R} = (X, R)$ be an abstract system. The (*interval order dimension*) *order dimension* $(idim(\mathfrak{R}))dim(\mathfrak{R})$ of (X, R) is the least cardinal λ such that there are λ (interval order) strict linear order extensions of R whose intersection is the transitive closure $\bar{R}$ of R.
Note that this definition coincides with the classical one when R is transitive.
The following theorem generalizes the well-known results of Hiraguchi [10], Ore [15] and Milner and Pouzet [14]. **Theorem 4.11.** Let $\mathfrak{R} = (X, R)$ be an abstract system where R is acyclic. Then the following statements are equivalent.

(a) The order dimension of $\mathfrak{R}$ is the least cardinal λ such that $\bar{R}$ is the intersection of λ strict linear orders.

(b) The order dimension of $\mathfrak{R}$ is the least cardinal λ such that there is a dominance-preserving embedding of $(X, \bar{R})$ into a strict direct product of λ strict linear orders.

(c) The order dimension of $\mathfrak{R}$ is the least cardinal λ such that there is a dominance-preserving embedding of $(X, \bar{R})$ into a direct product of λ linear orders.

Proof. **Step 1** $(dim(\mathfrak{R}) \geq spc(\mathfrak{R}))$. Suppose that $\mathfrak{R} = (X, R)$ has order dimension λ. Therefore, $\bar{R} = \bigcap_{i<\lambda} \hat{\mathcal{L}}_i$ where $\hat{\mathcal{L}}_i$ are strict linear orders on X. Let $\hat{\mathfrak{L}} = \{(X_i, \hat{\mathcal{L}}_i) \mid i < \lambda\}$. We define the map $f: (X, \bar{R}) \to (\tilde{X}, <_{S(\hat{\mathfrak{L}})}) = \odot \{(X, \hat{\mathcal{L}}_i) \mid i < \lambda$ by $f(x) = x_{i\,i<\lambda}$ where $x_i = x$ for all $i < \lambda$. Since the ordering $<_{S(\mathfrak{L})}$ is defined on X by $(x_i)_{i<\lambda} <_{S(\hat{\mathfrak{L}})} (y_i)_{i<\lambda}$ if and only if $x_i \hat{\mathcal{L}}_i y_i$ holds for all $i < \lambda$,
we have
$x\bar{R}y \Leftrightarrow (\forall i) x_i \hat{\mathcal{L}}_i y_i \Leftrightarrow (\forall i) x \hat{\mathcal{L}}_i y \Leftrightarrow f(x) <_{S(\hat{\mathfrak{L}})} f(y)$.

Step 2 $(spc(\mathfrak{R}) \geq dpc(\mathfrak{R}))$. To prove this fact, it suffices to show that the strict direct product $(\tilde{X}, <_{S(\hat{\mathfrak{L}})}) = \odot \{(X, \hat{\mathcal{L}}_i) \mid i < \lambda\}$ of the strict linear orders $\hat{\mathcal{L}}_i$ can be embedded into a direct product of linear orders. Indeed, let for each $i < \lambda$, $\mathcal{L}_i$ denote the ordering on $\tilde{X} = \prod_{i<\lambda} X^i$ defined by
$(x_j)_{j<\lambda} \mathcal{L}_i (y_j)_{j<\lambda}$ if and only if either $x_i \hat{\mathcal{L}}_i y_i$ or $x_i = y_i$ and $y_\gamma \hat{\mathcal{L}}_\gamma x_\gamma$ where
$\gamma = min\{\beta \mid x_\beta \neq y_\beta\}$.
Clearly, $\mathcal{L}_i$ is reflexive $(\{\beta \mid x_\beta \neq y_\beta\} = \emptyset)$, antisymmetric and transitive on $\tilde{X}$. We prove that $\mathcal{L}_i$ is also total on $\tilde{X}$. Suppose $x_i, y_i)$ $_i$ x_i y_i $[(x_i, y_i) \notin \hat{\mathcal{L}}_i \wedge (y_\gamma, x_\gamma) \notin \hat{\mathcal{L}}_\gamma, \gamma = min\{\beta \mid x_\beta \neq y_\beta\}]$ that $x_i \mathcal{L}_i y_i$ $[($ $\notin \hat{\mathcal{L}} \wedge$ $\neq$ $] \vee \mathcal{L}_i$ is false. Since $\mathcal{L}_i$ is total, it follows that
$= [(y_i \hat{\mathcal{L}}_i x_i)] \vee [(y_i \hat{\mathcal{L}}_i x_i) \wedge (x_\gamma \hat{\mathcal{L}}_\gamma x_\gamma)] \vee [(y_i = x_i) \wedge (x_\gamma \hat{\mathcal{L}}_\gamma x_\gamma)] = A \vee B \vee C.$

In all cases A, B and C we have $y_i \mathcal{L}_i x_i$. It follows that $\mathcal{L}_i$ is a linear order extension of $<_{S(\mathfrak{L})}$. We prove that $(\tilde{X}, <_{S(\mathfrak{L})})$ is embedded in the direct product $(\tilde{X}, <_{Q(\mathfrak{L})}) = \otimes\{(\tilde{X}, \mathcal{L}_i) \mid i < \lambda\}$, where $\mathfrak{L} = \{(X_i, \mathcal{L}_i) \mid i < \lambda\}$ and $\tilde{X} = \prod_{i<\lambda} \tilde{X}^i$. Let $\tilde{x} = (x_i)_{i<\lambda}$ where $x_i = x$ for all $i < \lambda$. We claim that $(\tilde{X}, <_{S(\mathfrak{L})})$ is embedded in the direct product $(\tilde{X}, <_{Q(\mathfrak{L})})$ by the mapping $f(\tilde{x}) = (\tilde{x}_i)_{i<\lambda}$ where $\tilde{x}_i = \tilde{x}$ for all $i < \lambda$. Indeed, if $\tilde{x} <_{S(\mathfrak{L})} \tilde{y}$, then $x_i \hat{\mathcal{L}}_i y_i$ and so $\tilde{x} \mathcal{L}_i \tilde{y}$ for all $i < \lambda$. Therefore, $f(\tilde{x}) <_{Q(\mathfrak{L})} f(\tilde{y})$.

Conversely, if $f(\tilde{x}) <_{Q(\mathfrak{L})} f(\tilde{y})$, then $\tilde{x} \neq \tilde{y}$ and $\tilde{x} \mathcal{L}_i \tilde{y}$ for all $i < \lambda$. Therefore, either $x_i \mathcal{L}_i y_i$ or $x_i = y_i$ for all $i < \lambda$. If $x_i = y_i$, then there is some $\gamma < \lambda$ such that $y_\gamma \mathcal{L}_\gamma x_\gamma$ and $\gamma = min\{\beta \mid x_\beta \neq y_\beta\}$. On the other hand, $\tilde{x} \mathcal{L}_i \tilde{y}$ for all $i < \lambda$ implies that $\tilde{x} \mathcal{L}_\gamma \tilde{y}$ and thus $x_\gamma \hat{\mathcal{L}}_\gamma y_\gamma$. Since $\hat{\mathcal{L}}_\gamma$ is transitive, $x_\gamma \hat{\mathcal{L}}_\gamma y_\gamma$ and $y_\gamma \hat{\mathcal{L}}_\gamma x_\gamma$ imply that $x_\gamma \hat{\mathcal{L}}_\gamma x_\gamma$, a contradiction to irreflexivity of $\hat{\mathcal{L}}_\gamma$. Therefore, $x_i \hat{\mathcal{L}}_i y_i$ for all $i < \lambda$. It follows that $\tilde{x} <_{S(\mathfrak{L})} \tilde{y}$. The last conclusion shows that
$\tilde{x} <_{S(\mathfrak{L})} \tilde{y} \Leftrightarrow f(\tilde{x}) <_{Q(\mathfrak{L})} f(\tilde{y})$.

Step 3 $(spc(\mathfrak{R}) \geq dim(\mathfrak{R}))$. Suppose that $spc(\mathfrak{R}) = \lambda$. By definition, λ is the least cardinal such that there is a dominance-preserving embedding f of $(X, \bar{R})$ into a direct product $(\tilde{X}, <_{Q(\mathfrak{M})}) = \otimes\{(X, \mathcal{M}_i) \mid i < \lambda\}$, where each $\mathcal{M}_i$ is a linear order, $\mathfrak{M} = \{\mathcal{M}_i \mid i < \lambda\}$ and $<_{Q(\mathfrak{M})}$ is defined by $(x_i)_{i<\lambda} \leq_{Q(\mathfrak{M})} (y_i)_{i<\lambda}$ if and only if $x_i \leq_{\mathcal{M}_i} y_i$ holds for all $i < \lambda$.
Then, by supposition we have
$x\bar{R}y \Leftrightarrow f(x) <_{Q(\mathfrak{M})} f(y)$.
If $f(x) = (x_i)_{i<\lambda}$, we write $f_i(x) = x_i$. Then, for each $i < \lambda$ define a linear order $\mathcal{C}_i$ on X by
$x\mathcal{C}_i y$ if and only if either $f_i(x) \neq f_i(y)$ and $f_i(x) \leq_{\mathcal{M}_i} f_i(y)$ hold or $f_i(x) = f_i(y)$ and $f_j(y) \leq_{\mathcal{M}_j} f_j(x)$, where

$$j = min\{k < \lambda \mid f_k(x) \neq f_k(y)\}.$$

We prove that

$$f(x) <_{Q(\mathfrak{M})} f(y) \Leftrightarrow (\forall i < \lambda)(x\mathcal{C}_i y)$$

.
Indeed, let $f(x) <_{Q(\mathfrak{M})} f(y)$, then $x \neq y$ and thus for all $i < \lambda$, we have $f_i(x) \leq_{\mathcal{M}_i} f_i(y)$ and $f_i(x) \neq f_i(y)$. Therefore, for all $i < \lambda$ we have $x\, \mathcal{C}_i y$. Hence,
$f(x) <_{Q(\mathfrak{M})} f(y) \Rightarrow (\forall i < \lambda)(x\mathcal{C}_i y)$.

Conversely, let $x\mathcal{C}_i y$ for all $i < \lambda$. Then, either (a) $f_i(x) \neq f_i(y)$ and $f_i(x) \leq_{\mathcal{M}_i} f_i(y)$ or (b) $f_i(x) = f_i(y)$ and $f_j(y) \leq_{\mathcal{M}_j} f_j(x)$, where $j = min\{k < \lambda \mid f_k(x) \neq f_k(y)\}$ hold. Suppose that $f_i(x) = f_i(y)$ for some $i < \lambda$. Then, $f_j(y) \leq_{\mathcal{M}_j} f_j(x)$, where $j = min\{k < \lambda \mid f_k(x) \neq f_k(y)\}$. Since $x\mathcal{C}_j y$ and $f_j(x) \neq f_j(y)$ we have $f_j(x) \leq_{\mathcal{M}_j} f_j(y)$. By the antisymmetry of $\leq_{\mathcal{M}_j}$ we have $f_j(x) = f_j(y)$ which is impossible by the definition of j. The last contradiction shows that for all $i < \lambda$ we have $[(f_i(x) \neq f_i(y)) \text{ and } (f_i(x) \leq_{\mathcal{M}_i} f_i(y))]$. It follows that
$(\forall i < \lambda)\ x\mathcal{C}_i y \Rightarrow (\forall i < \lambda)\ f_i(x) \neq f_i(y) \wedge f_i(x) \leq_{\mathcal{M}_i} f_i(y) \Rightarrow (\forall i < \lambda)(f_i(x) <_{\mathcal{M}_i} f_i(y)) \Rightarrow (\forall i < \lambda)(x_i <_{\mathcal{M}_i} y_i) \Rightarrow (x_i)_{i<\lambda} <_{Q(\mathfrak{M})} (y_i)_{i<\lambda} \Rightarrow f(x) <_{Q(\mathfrak{M})} f(y)$.
The last conclusion implies that,

$$f(x) <_{Q(\mathfrak{M})} f(y) \Leftrightarrow (\forall i < \lambda)(x\mathcal{C}_i y).$$

Therefore,

$$x\bar{R}y \Leftrightarrow f(x) <_{Q(\mathfrak{M})} f(y) \Leftrightarrow (\forall i < \lambda)(x\mathcal{C}_i y).$$

Since R is acyclic, the last implication implies that $\bar{R} = \cap_{i<\lambda}(\mathcal{C}_i \setminus \Delta)$, where for all $i < \lambda, \mathcal{C}_i \setminus \Delta$ is a strict linear order.
Because of the three steps above we conclude that $dim(\mathfrak{R}) = spc(\mathfrak{R}) = dpc(\mathfrak{R})$, and the proof is complete.
As an immediate consequence of Theorem 4.11, we have the following corollary which is the main result of
Corollary 4.12. Let $\mathfrak{F} = (X, \prec)$ be a poset. Then the following statements are equivalent.
(a) The order dimension of $\mathfrak{F}$ is the least cardinal λ such that $\prec$ is the intersection of λ strict linear orders.
(b) The order dimension of $\mathfrak{F}$ is the least cardinal λ such that there is an embedding of $(X, \prec)$ into a strictdirect product of λ strict linear orders.

(c) The order dimension of $\mathfrak{F}$ is the least cardinal λ such that there is an embedding of $(X,\prec)$ into a directproduct of λ linear orders.

An alternative definition of the interval order $\prec$ defined in X can be made by assigning to each element $x \in X$ an open interval $I_x = (a_x, b_x)$ of the real line, such that $x \prec y$ in X if and only if $b_x \leq a_y$. Such a collection of intervals is called an *interval representation* of $\prec$. Let $\lambda \in \aleph$ be a cardinal number and let $\mathcal{I} = (I_i)_{i<\lambda}$ be a family of interval orders. We denote by $\tilde{I}_i$ the interval order representation of each interval order I_i. Let (a_x^i, b_x^i) be an interval corresponding to $x \in X$ in the representation of $\tilde{I}_i$. With $x \in X$ we associate the box $\prod_{i<\lambda}(a_x^i, b_x^i) \subseteq \mathbb{R}^\lambda$. Each of these boxes is uniquely determined by its *upper extreme corner* $u_x = (b_x^i)_{i<\lambda}$ and its *lower extreme corner* $l_x = (a_x^i)_{i<\lambda}$. Such an assignment is called a *box embedding* of X. For the interval order dimension, the *box embedding* plays the role of the point embedding in $\mathbb{R}^\lambda$ introduced by Ore. The projections of a box embedding on each coordinate yields an interval order (see [5]).

To approach the interval orders analogue of the Hiraguchi [10], Ore [15] and Milner and Pouzet [14] results for posets, in a first step the concepts of direct product and strict direct product have to be generalized from linear orders to interval orders on X. The *direct product* of a family $\mathfrak{G} = \{(X_i, \preceq_i) \mid i < \lambda\}$ of strong interval orders is the Cartesian product $\prod_{i<\lambda} X^i$ equipped with the ordering $\preceq_{Q(\mathfrak{G})}$ defined by

$x \preceq_{Q(\mathfrak{G})} y$ if and only if either $b_x^i \leq a_y^i$ or $a_x^i = a_y^i, b_x^i = b_y^i$ holds for all $i < \lambda$.

The *strict direct product* of a family $\mathfrak{G} = \{(X_i, \prec_i) \mid i < \lambda\}$ of interval orders is the Cartesian product $\prod_{i<\lambda} X^i$ equipped with the ordering $\prec_{S(\mathfrak{G})}$ defined by $x \prec_{S(\mathfrak{G})} y$ if and only if $b_x^i \leq a_y^i$ holds for all $i < \lambda$.

Definition 4.3. Let $\mathfrak{P} = (X, R)$ be an abstract system. (i)We call $idpc(\mathfrak{P})$, the least cardinal λ such that there is a box embedding of $(X, \bar{R})$ into a direct product of λ strong interval orders. (ii) We call $spc(\mathfrak{P})$ the least cardinal λ such that there is a box embedding of $(X, \bar{R})$ into a direct product of λ interval orders.

Theorem 4.13. Let $\mathfrak{P} = (X, R)$ be an abstract system where R is acyclic. Then the following statements are equivalent.

(a) The interval order dimension of $\mathfrak{P}$ is the least cardinal λ such that $\bar{R}$ is the intersection of λ interval orders.

(b) The interval order dimension of $\mathfrak{P}$ is the least cardinal λ such that there is a box embedding of $(X, \bar{R})$ into a strictdirect product of λ interval orders.

(c) The interval order dimension of $\mathfrak{P}$ is the least cardinal λ such that there is a box embedding of $(X, \bar{R})$ into a direct product of λ strong interval orders.

Proof. **Step 1** $(idim(\mathfrak{P}) \geq ispc(\mathfrak{P}))$. Suppose that $\mathfrak{P} = (X, R)$ has interval order dimension λ. Therefore, $\bar{R} = \bigcap_{i<\lambda} \prec_i$ where $\prec_i$ are interval orders on X. Let $\mathcal{I} = \{I_x^i \mid x \in X\}$, where $I_x^i = (a_x^i, b_x^i)$ be an interval representation of $\prec_i$. Let also $\tilde{X} = \prod_{i<\lambda} X^i$ and $\mathfrak{H} = \{\prec_i \mid i < \lambda\}$. We define the map $f: (X, \bar{R}) \to (\tilde{X}, \prec_{S(\mathfrak{H})}) = \odot\{(X, \prec_i) \mid i < \lambda\}$ by $f(x) = \prod_{i<\lambda}(a_x^i, b_x^i)$.

The ordering $\prec_{S(\mathfrak{H})}$ is defined by $f(x) \prec_{S(\mathfrak{H})} f(y)$ if and only if $b_x^i \leq a_y^i$ holds for all $i < \lambda$.

Therefore,

$x\bar{R}y \Leftrightarrow (\forall i < \lambda)[b_x^i \leq a_y^i] \Leftrightarrow f(x) \prec_{S(\mathfrak{H})} f(y)$.

Step 2 $(ispc(\mathfrak{P} \geq idpc(\mathfrak{P}))$. To show this fact, it suffices to show that the strict direct product $(\tilde{X}, \prec_{S(\mathfrak{H})}) = \odot$ $\{(X, \prec_i) \mid i < \lambda\}$ can be box embedded into a direct product of strong interval orders. Indeed, let $\mathcal{I} = \{I_x^i \mid x \in X\}$, where for each $i < \lambda, I_x^i = (a_x^i, b_x^i)$ be an interval representation of $\prec_i$. For each $i < \lambda$, define the ordering $\sqsubseteq_i$ on X by $(x_j)_{j<\lambda} \sqsubseteq_i (y_j)_{j<\lambda}$ if and only if either (i) $b_x^i \leq a_y^i$ or (ii) $a_x^i = a_y^i, b_x^i = b_y^i$ and

where $k = min\{\mu \mid a_x^\mu \neq a_y^\mu$ or $b_x^\mu \neq b_y^\mu\}$].

Clearly, for all $i, \sqsubseteq_i$ is an extension of $\prec_{S(\mathfrak{H})}$. Since the reals satisfy the law of trichotomy we conclude that for each $i < \lambda, \sqsubseteq_i$ is a strong interval order. We show that $(\tilde{X}, \prec_{S(\mathfrak{H})})$ is box embedded in the direct product $(\tilde{\tilde{X}}, \prec_{Q(\mathfrak{D})}) = \otimes$

$\{(\tilde{X}, \sqsubseteq_i) \mid i < \lambda\}$, where $\mathfrak{D} = \{(\tilde{X}, \sqsubseteq_i) \mid i < \lambda\}$ and $\tilde{\tilde{X}} = \prod_{i<\lambda} \tilde{X}^i$. Let $\tilde{x} = (x_i)_{i<\lambda}$, where $x_i = x$ for all $i < \lambda$. By definition, the ordering $\prec_{Q(\mathfrak{D})}$ is defined $(\tilde{x}_j)_{j<\lambda} \prec_{Q(\mathfrak{D})} (\tilde{y}_j)_{j<\lambda}$ if and only if $(\tilde{x}_j)_{j<\lambda} \sqsubseteq_i (\tilde{y}_j)_{j<\lambda}$ holds for all $i < \lambda$.

Let f be the mapping $f(\tilde{x}) = (\tilde{x}_i)_{i<\lambda}$, where $\tilde{x}_i = \tilde{x}$ for all $i < \lambda$. Clearly, there holds the following implication:
$\tilde{x} <_{S(\mathfrak{D})} \tilde{y} \Leftrightarrow (\forall i < \lambda)[b_x^i \leq a_y^i] \Rightarrow (\forall i \quad \lambda)[\tilde{\ } \quad \tilde{\ }] \quad f(\tilde{\ }) \quad f(\tilde{\ }) < \quad x \sqsubseteq_i y \Leftrightarrow x <_{Q(\mathfrak{D})} y$.

Conversely, if $f(\tilde{x}) <_{Q(\mathfrak{D})} f(\tilde{y})$, then $\tilde{x} \sqsubseteq_i \tilde{y}$ for all $i < \lambda$. Therefore, for all $i < \lambda$
$[b_x^i \leq a_y^i] \vee [(a_x^i = a_y^i, b_x^i = b_y^i) \wedge (b_y^k \leq a_x^k$ where $k = min\{\mu \mid a_x^\mu \neq a_y^\mu \text{ or } b_x^\mu \neq b_y^\mu)]$

Suppose that $a_x^i = a_y^i$ and $b_x^i = b_y^i$ for some $i < \lambda$. Then, there is some k such that $b_y^k \leq a_x^k$. On the other hand, since $\tilde{x} \sqsubseteq_k \tilde{y}$ and $a_y^k < b_y^k \leq a_x^k < b_x^k (a_x^k \neq a_y^k$ and $b_x^k \neq b_y^k)$, we have that $b_x^k \leq a_y^k$. But then, $b_y^k \leq a_x^k < b_x^k \leq a_y^k$ implies $b_y^k < a_y^k$ which is impossible. The last contradiction shows that for all $i < \lambda$ there holds $b_x^i \leq a_y^i$, which implies that $\tilde{x} <_{S(\mathfrak{D})} \tilde{y}$. Therefore,
$\tilde{x} <_{S(\mathfrak{D})} \tilde{y} \Leftrightarrow f(\tilde{x}) <_{Q(\mathfrak{D})} f(\tilde{y})$.

Step 3 $(idpc(\mathfrak{P}) \geq idim(\mathfrak{P}))$. Suppose that $idpc(\mathfrak{P}) = \lambda$. Then, λ is the least cardinal such that there is a box embedding f of $(X, \bar{R})$ into a direct product $(\tilde{X}, <_{Q(\mathfrak{T})}) = \otimes\{(X, \trianglelefteq_i) \mid i < \lambda\}$ of strong interval orders $\{\trianglelefteq_i \mid i < \lambda\} = \mathfrak{T}$. By definition, $\tilde{X} = \prod_{i<\lambda} X^i$. On the other hand, if for all $i < \lambda$, $\mathfrak{I}_x^i = (\alpha_x^i, \beta_x^i)$ is an interval representation of $\trianglelefteq_i$, then the ordering $<_{Q(\mathfrak{T})}$ is defined by $(x_i)_{i<\lambda} <_{Q(\mathfrak{T})} (y_i)_{i<\lambda}$ if and only if $\beta_x^i \leq \alpha_y^i$ holds for all $i < \lambda$.
Then, by definition, we have
$x\bar{R}y \Leftrightarrow f(x) <_{Q(\mathfrak{T})} f(y)$.

If $f(x) = (x_j)_{j<\lambda}$, then for each $i < \lambda$ we define the ordering $\ll_i$ on X by: $x \ll_i y$ if and only if either $\beta_x^i \leq \alpha_y^i$ or $\alpha_x^i = \alpha_y^i, \beta_x^i = \beta_y^i$ and $\beta_y^k \leq \alpha_x^k$, where $k = min\{\mu \mid a_x^\mu \neq a_y^\mu \text{ or } b_x^\mu \neq b_y^\mu)$.

Clearly, each $\ll_i$ is a strong interval $f(\ )$ $f(\ )$ order extension of $\trianglelefteq_i$. We prove that
$x <_{Q(\mathfrak{T})} y \Leftrightarrow (\forall i \quad \lambda)(< \quad x \ll_i y)$.

Indeed, let $f(x) <_{Q(\mathfrak{T})} f(y)$. Then, $x \neq y$ and for any $i < \lambda$ there holds $\beta_x^i \leq \alpha_y^i$ and so $x \ll_i y$ for all $i < \lambda$. Conversely, let $x \ll_i y$ for all $i < \lambda$. Then, either
(a) $[(a_x^i \neq a_y^i) \vee (b_x^i \neq b_y^i)] \wedge (\beta_x^i \leq \alpha_y^i)$
or
(b) $[(a_x^i = a_y^i) \wedge (b_x^i = b_y^i)] \wedge [(\beta_y^j \leq \alpha_x^j)$, where $j = min\{k < \lambda \mid a_x^k \neq a_y^k \text{ or } b_x^k \neq b_y^k]$

Suppose that $a_x^i = a_y^i$ and $b_x^i = b_y^i$ for some $i < \lambda$. Then, $\beta_y^j \leq \alpha_x^j$ where j has the meaning above mentioned. On the other hand, $\beta_y^j \leq \alpha_x^j$ implies that $a_x^j \neq a_y^j$ or $b_x^j \neq b_y^j$. Since $x \ll_j y$ we have that $\beta_x^j \leq \alpha_y^j$. It follows that $\beta_y^j < \alpha_y^j (\alpha_x^j < \beta_x^j$ which is impossible. The last conclusion shows $(\forall i < \lambda)(\quad) \quad (\forall i < \lambda)(\beta \leq {}^i) \quad (\ ) < (\ ) \quad f(\ )$ that for all $i < \lambda$ we have that the case (a) holds. Therefore,
$x \ll_i y \Rightarrow \quad {}_x^i \quad \alpha_y \Rightarrow x_i \ {}_{i<\lambda} \quad {}_{Q(\mathfrak{T})} \ y_i \ {}_{i<\lambda} \Rightarrow \quad x \quad {}_{Q(\mathfrak{T})}$
$f(y)$.

Therefore, by combining the previous implications, we get
$(\forall i < \lambda)(x \ll_i y) \Leftrightarrow f(x) <_{Q(\mathfrak{T})} f(y)$.

Finally, by $x\bar{R}y \Leftrightarrow f(x) <_{Q(\mathfrak{T})} f(y)$, we have that
$x\bar{R}y \Leftrightarrow f(x) <_{Q(\mathfrak{T})} f(y) \Leftrightarrow (\forall i < \lambda)(x \ll_i y)$.

Since R is acyclic, the last implication implies that $\bar{R} = \cap_{i<\lambda}(\ll_i \setminus \Delta)$, where for all $i < \lambda$, $\ll_i \setminus \Delta$ is an interval order. Because of the three steps above we conclude that $dim(\mathfrak{R}) = spc(\mathfrak{R}) = dpc(\mathfrak{R})$, and the proof is complete.

The following corollary is an immediate consequence of Theorem 4.13.

Corollary 4.14. Let $\mathfrak{G} = (X, \prec)$ be a poset. Then the following statements are equivalent.

(a) The interval order dimension of $\mathfrak{G}$ is the least cardinal λ such that $\prec$ is the intersection of λ interval orders.

(b) The interval order dimension of $\mathfrak{G}$ is the least cardinal λ such that there is a box embedding of $(X, \prec)$ into a strict direct product of λ interval orders.

(c) The interval order dimension of $\mathfrak{G}$ is the least cardinal λ such that there is a box embedding of $(X, \prec)$ into a direct product of λ strong interval orders.

Let T be a triangle ABC. Denote $\kappa(T) = A$and $\pi(T) = BC$. Let L_1andL_2 be two distinct parallel lines. A point-interval graph or*PI* graph is the intersection graph of a family of triangles ABC, such that $A$is on$L_1$and$BC$is onL_2.Except for the definition we gave in the introduction, the linear-interval order can also be defined as follows: An acyclic binary relation R is a linear-interval order if there is such a triangle $T_x$for each element$x \in X$, and $(y,x) \in R$ if and only if$T_y$lies completely to the left ofT_x. In fact, the ordering of the apices $\kappa(T_x) = x$ of the triangles gives the linear order L, and the bases $\pi(T_x) = (a_x, b_x)$ of the triangles give an interval representation of the interval order P for which $\bar{R} = L \cap P$. As usual, the left and right extreme points of an interval I_x are denoted by $a_x$andb_x ,respectively. When $a_x = b_x = x$, we say that I_x is trivial. Let $\mathbb{R}^\lambda$ be the cartesian product of λ many copies of $\mathbb{R}$. A linear-interval point γ is the set $\prod_{i<\lambda} I_i$ where $I_i \subset \mathbb{R}$for all$i < \lambda$(notice that in this definition it is allowed that I_i be trivial). With $x \in X$ we associate the box $\prod_{i<\lambda}(a_x^i, b_x^i) \subseteq \mathbb{R}^\lambda$. This assignment is called a *linear interval box embedding*of X.

For the linear-interval order dimension, the linear-interval box embedding plays the role of the point embedding into $\mathbb{R}^\lambda$ introduced by Ore. The projections of a linear-interval box embedding on each coordinate yield a linear order or an interval order.

To approach the linear-interval orders analogue of the Hiraguchi [10], Ore [15] and Milner and Pouzet [14] results for posets, in a first step the concepts of direct product and strict direct product must be generalized from linear orders and interval orders to linear-interval orders on X. The *direct product* of a family $\mathfrak{G} = \{(X_i, \leq_i) \mid i < \lambda\}$of strong linear-interval orders is the Cartesian product $\prod_{i<\lambda} X^i$ equipped with the ordering $\leq_{Q(\mathfrak{G})}$ defined by $x \leq_{Q(\mathfrak{G})} y$ if and only if either $b_x^i \leq a_y^i$ or $a_x^i = a_y^i, b_x^i = b_y^i$ holds for all $i < \lambda$.

The *strict direct product* of a family $\mathfrak{G} = \{(X_i, \prec_i) \mid i < \lambda\}$ of linear-interval orders is the Cartesian product $\prod_{i<\lambda} X^i$ equipped with the ordering $\prec_{S(\mathfrak{G})}$ defined by $x \prec_{S(\mathfrak{G})} y$ if and only if $b_x^i \leq a_y^i$ holds for all $i < \lambda$.

Definition 4.4. Let$\mathfrak{P} = (X, R)$be an abstract system. (i) We call$lidpc(\mathfrak{P})$,the least cardinal λsuch that there is a linear-interval box embedding of $(X, \bar{R})$ into a direct product of λstrong linear-interval orders. (ii) We call $lispc(\mathfrak{P})$, the least cardinalλ such that there is a linear-interval box embedding of $(X, \bar{R})$into a direct product ofλ linearinterval orders.

The following theorem generalizes Theorem 4.11 and Theorem 4.13. The prove is omitted since it follows the same scheme.

Theorem 4.15. Let $\mathfrak{P} = (X, R)$ be an abstract system where R is acyclic. Then the following statements are equivalent.

(a) The (λ, μ)-linear-interval order dimension of $\mathfrak{P}$ is the least cardinal λ such that $\bar{R}$ is the intersection of λ linearinterval orders which μ of them are not linear orders.

(b) The (λ, μ)-linear-interval order dimension of $\mathfrak{P}$ is the least cardinal λ such that there is a linear-interval embeddingof $(X, \bar{R})$ into a strict direct product of λ linear interval orders, of which μ are not strict linear orders.

(c) The (λ, μ)-linear-interval order dimension of $\mathfrak{P}$ is the least cardinal λ such that there is a strong linearintervalembedding of $(X, \bar{R})$into a direct product of λ strong linear-interval orders whichμ of them are not linear *orders*.

The following corollary is an immediate consequence of Theorem 4.17.

Corollary 4.16. Let $\mathfrak{G} = (X, \prec)$ be a poset. Then the following statements are equivalent.

(a) The (λ, μ)-linear-interval order dimension of $\mathfrak{G}$ is the least cardinal λ such that $\prec$ is the intersection of λ linearinterval orders whichμ of them are not linear orders.

(b) The (λ, μ)-linear-interval order dimension of $\prec$ is the least cardinal λ such that there is a linear-interval embeddingof$(X, \prec)$ into a strict direct product of λ linear-interval orders whichμ of them are not strict linear orders.

(c) The (λ, μ)-linear-interval order dimension of $\mathfrak{G}$ is the least cardinal λ such that there is a strong linearintervalembedding of $(X, \prec)$ into a direct product of λ strong linear-interval orders which μ of them are not linear orders.

Using the previous approach for linear-interval orders, we can define in a similar way the notion of (strong) linearsemiorder box embedding. The only difference is that a semiorder is a poset whose elements correspond to unit length intervals.

Definition 4.5. Let $\mathfrak{P} = (X, R)$be an abstract system. (i) We call$sidpc(\mathfrak{P})$,the least cardinalλ such that there is a linear-semiorder box embedding of $(X, \bar{R})$ into a direct product of λstrong linear-semiorders. (ii) We call $sispc(\mathfrak{P})$, the least cardinal λ such that there is a linear-semiorder box embedding of $(X, \bar{R})$ into a direct product ofλ linearsemiorders.

The following two theorems are proved in a similar way to the proof of Theorems 4.11 and 4.13, by using Theorem 3.8, Theorem 3.9 and Definition 4.5.

Theorem 4.17. Let $\mathfrak{P}$=(X,R) be an abstract system where R is acyclic. Then the following statements are equivalent.

(a) The (λ,)-linear-semiorder dimension of $\mathfrak{P}$ is the least cardinal λ such that R is the intersection of λ linearsemiorders whichμ of them are not linear orders.

(b) The (λ,μ)-linear-semiorder dimension of $\mathfrak{P}$ is the least cardinal λ such that there is a linear-semiorder embeddingof(X,R) into a strict direct product ofλ linear-semiorders which μ of them are not strict linear orders. (c) The (λ,μ)-linear-semiorder dimension of $\mathfrak{P}$ is the least cardinal λ such that there is a strong linearsemiorderembedding of (X,R) into a direct product of λ strong linear-semiorders which μ of them are not linear orders.

The following corollary is an immediate consequence of Theorem 4.17.

Corollary 4.18. Let $\mathfrak{G}$=(X,$\prec$) be a poset. Then the following statements are equivalent.

(a) The (λ,)-linear-semiorder dimension of $\mathfrak{G}$ is the least cardinal λ such that $\prec$ is the intersection ofλlinearsemiorders which μ of them are not linear orders.

(b) The (λ,μ)-linear-semiorder dimension of $\prec$ is the least cardinal λ such that there is a linear-semiorder embeddingof(X,$\prec$) into a strict direct product of λ linear-semiorders whichμ of them are not strict linear orders. (c) The (λ,μ)-linear-interval order dimension of $\mathfrak{G}$ is the least cardinal λ such that there is a strong linearsemiorderembedding of (X,$\prec$) into a direct product of λ strong linear-interval orders which μ of them are not linear orders.

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