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Linear and Nonlinear Functionsof Norm Minimizing Estimation in Set Indexed Stochastic Processes

ARTHUR YOSEF
Tel Aviv-Yaffo Academic College, Israel

Abstract: This article discusses the norm minimizing estimation of a set indexed stochastic process using another set indexed stochastic process as a reference. The collection A, which generates Borel sets of a topological space, plays a critical role in the selection of the indexing collection. We introduce the norm minimizing estimation of the set indexed stochastic process in terms of linear and nonlinear functions of another set indexed stochastic process. We also prove the orthogonality principle under certain assumptions. Additionally, we define the inner product and equivalence relation on square integrable set indexed stochastic processes and introduce an inner product and a norm on the quotient set. We present Theorem 2, which provides the necessary conditions for the set indexed norm minimizing estimation of the set indexed stochastic process using a constant set indexed process, a linear function of set indexed process, and a nonlinear function of set indexed process.

Keywords: set indexed stochastic processes, compact set collection, norm minimizing estimation, linear function, nonlinear function, orthogonality principle, Borel sets, inner product, equivalence relation, constant set indexed process, square integrable stochastic processes.

INTRODUCTION

In this article, we present the norm minimizing estimation method of a set indexed stochastic process by another set indexed stochastic process, when the set index $\mathbf{A}$is a compact set collection on a topological space
T. The choice of the collection A is critical: it must be sufficiently rich in order to generate the Borel sets of$T$, but small enough to ensure the existence of a continuous process defined on $\mathbf{A}$.

We introduce a norm minimizing estimation of a set indexed stochastic process $Y = \{Y_A : A \in \mathbf{A}\}$in terms of another set indexed stochastic process$X = \{X_A : A \in \mathbf{A}\}$by a linear and a nonlinear function of$X$. We prove with some assumptions that a set indexed of$Y$by linear function of set indexed process is aX +$b$when $a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$ and $b = 0$, by nonlinear function of set indexed process is $\lim_{A \nearrow T} E[Y_A | X_A]$.

In addition, we present the orthogonality principle. We prove with some assumptions that a set indexed norm minimizing estimation of $Y$is$X$if and only if $Y - aX, X$are orthogonal, when$a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$.

Preliminaries

In the study, processes are indexed by an indexing collection $\mathbf{A}$ (see [IvMe]) of compact subsets of a locally metric and separable space$T$. We use the definition of $\mathbf{A}$ and notation from [IvMe] and all this section comefrom there:

Let (T, τ) be a non-void sigma-compact connected topological space. A nonempty class $\mathbf{A}$ of compact, connected subsets of $T$is called an indexed collection if it satisfies the following:

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$\emptyset \in \mathbf{A}$. In addition, there is an increasing sequence (B_n) of sets in $\mathbf{A}$ s.t.$T = \bigcup_{n=1}^{\infty} B_n^{\circ}$.
$\mathbf{A}$is closed under arbitrary intersections and if $A, B \in \mathbf{A}$ are nonempty, then $A \cap B$ is nonempty. If (A_i) is an increasing sequence in $\mathbf{A}$and if there exists $n$such that$A_i \subseteq B_n$for every$i$, then$\overline{\bigcup_i A_i} \in \mathbf{A}$.
$\sigma(\mathbf{A}) = \mathbf{B}$ where$\mathbf{B}$is the collection of Borel sets of$T$.
We will require other classes of sets generated by$\mathbf{A}$. The first is$\mathbf{A}(\mathbf{u})$, which is the class of finite unions of sets in$\mathbf{A}$. We note that $\mathbf{A}(\mathbf{u})$is itself a lattice with the partial order induced by set inclusion. Let $\mathbf{C}$consists of all the subsets of $T$of the form
$C = A \quad B, A \in \mathbf{A}, B \in \mathbf{A} \ \mathbf{u} \backslash \qquad (\ )$.
A set-indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ is additive if ithas an (almost sure) additive extension to$\mathbf{C}$ X : $_{\emptyset} = 0$and if$C, C_1, C_2 \in \mathbf{C}$ with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$ then almost surely$X_C = X_{C_1} + X_{C_2}$. In particular, if $C \in \mathbf{C}$and $C = A \backslash \bigcup_{i=1}^{n} A_i$, $A, A_1, \ldots, A_n \in \mathbf{A}$ then almost surely
$X_C = X_A - \sum_{i=1}^{n} X_{A \cap A_i} + \sum_{i<j} X_{A \cap A_i \cap A_j} - \ldots + (-1)^n X_{A \cap \bigcap_{i=1}^{n} A_i}$.

We shall always assume that our stochastic processes are additive. We note that a process with an (almost sure) additive extension to$\mathbf{C}$also has an (almost sure) additive extension to $\mathbf{C}(\mathbf{u})$.

Norm minimizing estimation in set indexed stochastic processes

Definition 1.

Let$A = \{A_n\}$be an increasing sequence in$\mathbf{A}$. We write $A_n \uparrow T$(or, in short notation$A \uparrow T$) if $A_n \neq T$for all $n$and$\overline{\bigcup_n A_n} = T$.
We write $A \nearrow T$ if $A_n \uparrow T$ for all an increasing sequence$\{A_n\}$ in $T^{\uparrow} = \{\{A_n\} : A_n \uparrow T\}$
We introduce the estimation of a set indexed stochastic process $Y = \{Y_A : A \in \mathbf{A}\}$in terms of another set indexed stochastic process$X = \{X_A : A \in \mathbf{A}\}$. Throughout this analysis, the optimality criterion will be the minimization of the norm value of the estimation.Let$X = \{X_A : A \in \mathbf{A}\}$, $Y = \{Y_A : A \in \mathbf{A}\}$be a squareintegrable set indexed stochastic processes. We define the inner product:
$\langle X, Y \rangle = \exists \lim_{A \nearrow T} Cov(X_A, Y_A)$
(In another words, for all $\{A_n\} \in T^{\uparrow}$the limits are existing and equal). Easy to see that
$\|X\| = \sqrt{\langle X, X \rangle}$
is a semi-norm.
We define the equivalence relation on square-integrable set indexed stochastic processes:
$\|X - Y\| = 0 \quad \Leftrightarrow \quad X \approx Y$
We denote the quotient setby$H$ (In another words, $H = \{[X]_{\approx} : X \in L^2(A)\}$ then $[X]_{\approx}$is an equivalence class). Now, we can define an inner product and a norm on$H$:
$\langle X, Y \rangle_H = \langle X, Y \rangle$and$\|X\|_H = \sqrt{\langle X, X \rangle_H}$
for all $X, Y \in H \ \ (X \in [X]_{\approx}, Y \in [Y]_{\approx})$

Definition 2.

Let X, Ybe a random variables with finite variance. We say that estimation of $Y$is$X$if $E[(Y - X)^2]$is minimal (see [Pa]).
Let$X, Y \in H$. We say that set indexed norm minimizing estimation of $Y$is$X$if $\|X - Y\|_H^2$is minimal.

Theorem 2. Let$X, Y \in H$and$\lim_{A \nearrow T} E[X_A] = \lim_{A \nearrow T} E[Y_A] = 0$.
Set indexed norm minimizing estimation of$Y$by constant set indexed process$X = c$when

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$c = 0$.
Set indexed norm minimizing estimation of Y by linear function of set indexed process is $aX + b$ when $a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$ and $b = 0$.
Set indexed norm minimizing estimation of Y by nonlinear function of set indexed process is $\lim_{A\nearrow T} E[Y_A|X_A]$.
Proof.
Define $g(c) = \|Y - c\|_H^2 = \lim_{A\nearrow T} E[Y_A - c]^2$. Clearly, $g(c)$ is minimum if $g'(c) = -2\lim_{A\nearrow T} E[Y_A - c] = 0$. Then $c = 0$ for $A \nearrow T$.
For a given a, set indexed norm minimizing estimation of $Y - aX$ is a constant set indexed process. Then from (a) we get:
$b = 0$.
Define

$$g(a) = \|Y - aX\|_H^2 = \lim_{A\nearrow T} E[Y_A - aX_A]^2 = .$$
$$= \langle Y,Y\rangle_H - 2a\langle X,Y\rangle_H + \langle X,X\rangle_H a^2$$

Clearly, $g(a)$ is minimum if $g'(a) = 0$.
Then
$a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$.
We must find the function $g(x)$ such that $\|Y - g(X)\|_H^2 = \lim_{A\nearrow T} E[(Y_A - g(X_A))^2]$ is minimum.
$\lim_{A\nearrow T} E[(Y_A - g(X_A))^2] = \lim_{A\nearrow T} \iint_{\Re^2} [y - g(x)]^2 \, dF_{Y_A,X_A}(x,y)$.
But
$F_{Y_A,X_A}(x,y) = F_{X_A}(x) F_{Y_A|X_A}(y)$,
then

$$\|Y - g(X)\|_H^2 = \lim_{A\nearrow T} \int_{-\infty}^{\infty} dF_{X_A}(x) \int_{-\infty}^{\infty} [y - g(x)]^2 dF_{Y_A|X_A}(y)$$

The integrands above are positive. Hence $\|Y - g(X)\|_H^2$ is minimum if the inner integral is minimum for every x. Hence it is minimum if $g(x)$ is constant. Then from (a) we get: $g(x) = \lim_{A\nearrow T} \int_{-\infty}^{\infty} y dF_{Y_A|X_A}(y) = \lim_{A\nearrow T} E[Y_A|x]$. □
Theorem 3. (The orthogonality principle) Let $X, Y \in H$ and $\lim_{A\nearrow T} E[Y_A] = \lim_{A\nearrow T} E[X_A] = 0$. $\|Y - aX\|_H^2$ is minimal if and only if $\langle Y - aX, X\rangle_H = 0$.
(In other words, set indexed norm minimizing estimation of Y by linear function of set indexed process is aX when $a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$ and $b = 0$ if and only if $Y - aX \perp X$).
(Note: Two random variables are called orthogonal if $E[XY] = 0$. We shall use the notation $X \perp Y$ to indicate that X, Y are orthogonal).
Proof.
Based on Theorem 2(b), set indexed norm minimizing estimation of Y by linear function of set indexed process is $aX + b$ when $a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$ and $b = 0$.
If we define
$g(a) = \|Y - aX\|_H^2 = \lim_{A\nearrow T} E[Y_A - aX_A]^2$ then
$g(a) = \lim_{\nearrow} E[Y_A - aX_A]^2 = \lim_{\nearrow} E[Y_A^2] - 2a \lim_{\nearrow}$ $A\ T$ $A\ T$ $A\ T$

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Clearly, $g(a)$ is minimum if $\lim_{A\nearrow T} g'(a) = 0$.
Then $\lim_{A\nearrow T} E[(Y_A - aX_A)X_A] = 0$. □

Theorem 4. Let $X, Y \in H$. If the random variables X_A, Y_A are Gaussian (or X, Y are Brownian motions [BoSa, Da, Du, Fr, ReYo]) for all $A \in \mathbf{A}$ and $\lim_{A\nearrow T} E[Y_A] = \lim_{A\nearrow T} E[X_A] = 0$ then
Set indexed norm minimizing estimation of Y by linear function of set indexed process is equal to set indexed norm minimizing estimation of Y by nonlinear function of set indexed process.
(In other words, $\lim_{A\nearrow T} E[Y_A|X_A] = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H} \lim_{A\nearrow T} E[X_A]$).

Proof.

The random variables X_A, Y_A are normal for all $A \in \mathbf{A}$ and $\lim_{A\nearrow T} E[Y_A] = \lim_{A\nearrow T} E[X_A] = 0$.
From Theorem 2(b) we get that, set indexed norm minimizing estimation of Y by linear function of set indexed process is $aX + b$ when $a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$ and $b = 0$. Then $X_A, Y_A - aX_A$ are uncorrelated since $\langle Y - aX, X\rangle_H = 0$. But X_A, Y_A are normal then $X_A, Y_A - aX_A$ are independent.
Then
$\lim_{A\nearrow T} E[Y_A - aX_A|X_A]$ and on the other hand $\lim_{A\nearrow T} E[Y_A - aX_A|X_A] = \lim_{A\nearrow T} E[Y_A|X_A] - a \lim_{A\nearrow T} E[X_A|X_A] = \lim_{A\nearrow T} E[Y_A|X_A] - a \lim_{A\nearrow T} X_A$ and from that we get,
$\lim_{A\nearrow T} E[Y_A|X_A] = a \lim_{A\nearrow T} E[X_A]$ when $a = \frac{\langle X,Y\rangle_H}{\langle X,X\rangle_H}$. □

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