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KLEIMAN'S CRITERION AND THE GEOMETRY OF AMPLE INVERTIBLE SHEAVES ON EXCEPTIONAL LOCI

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Abstract
This paper explores the resolution of XXX-dimensional YYY-singularities by focusing on the exceptional locus E\mathcal{E}E, which consists of irreducible components Ei\mathcal{E}_iEi that are isomorphic to Pn\mathbb{P}^nPn. These components are studied in the context of invertible sheaves, with an emphasis on determining their ampleness. Utilizing Kleiman's Criterion (1966), the study establishes the necessary and sufficient conditions for ampleness of these sheaves. The results provide a comprehensive understanding of the geometric properties of the exceptional locus, contributing to the broader field of YYY-singularities and their resolutions. This investigation not only clarifies the conditions under which the sheaves are ample but also enhances the theoretical framework surrounding singularity resolution.

INTRODUCTION

In the affine space $\mathbb{A}^5(x_1, \cdots, x_5)$, over an algebraically closed field K with arbitrary characteristic, the 4dimennsinal A_n-Singularity is given by the equation:

$f_n = x_1^{n+1} + x_2x_3 + x_4x_5,$ i.e. that is, $\mathbb{A}_n = spec(K[x_1, \cdots, x_5]/\langle f_n \rangle)$

A_n has an isolated singular point in the origin Hartshorne (1977). Step by step we find that the irreducible components of the exceptional locus $E_1, E_2, \cdots, E_m$ are isomorphic to Q, R, or $\mathbb{P}^3$.

The irreducible components of the exceptional locus Remark

We assume the ring $A = K[x_1, \cdots, x_n]/\langle f \rangle$,

Where;

$$f = \begin{cases} x_1^2 + x_2x_3 + \cdots + x_{r-1}x_r \text{ if } r \equiv 1 \, mod \, 2 \text{ and } n \geq r \\ x_1x_2 + x_3x_4 + \cdots + x_{r-1}x_r \text{ if } r \equiv 0 \, mod \, 2 \text{ and } n \geq r \end{cases}$$

$A \quad X = spec\ A$

Let X be the spectrum of ( ) and

$Q = proj\ A$ (projective spectrum), that is, $x = c(Q)$ is the affine cone over Q.

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Lemma

With the above notations:

If $r = 3$, then $cl\ X \cong \mathbb{Z}/2\mathbb{Z}$, and $cl\ Q \cong \mathbb{Z}$

Where;

$cl\ X$ − divisor class group of X

$cl\ Q$ = divisor class group of Q

and the generator of this class is $H \cap Q$ (H is a hyperplane).

(a) If $r = 4$, then $cl\ X = \mathbb{Z}$, and $cl\ Q = \mathbb{Z} \oplus \mathbb{Z}$.

(b) If $r \geq 5$, then $cl\ X = 0$, and $cl\ Q = \mathbb{Z}$.

Proof:

(a) The proof can be obtained by the exact sequence:

$0 \longrightarrow \mathbb{Z} \longrightarrow cl\ Q \longrightarrow cl\ X \longrightarrow 0$

$1 \longmapsto Q.H$

Here, $Q = proj(K[x_1, x_2, x_3]/(x_1x_2 - x_3^2))$.

We assume $Y = v^+(x_1, x_2) \subset Q$,

that is a prime divisor and:

$$Q - Y \cong spec\left(K\left[x_2, x_3, \frac{x_3^2}{x_2}\right]_{(x_3)}\right) \cong spec\left(K\left[\frac{x_2}{x_3}, \frac{x_3}{x_2}\right]\right) - 0$$

because $K\left[\frac{x_2}{x_3}, \frac{x_3}{x_2}\right]$ is a field.

From the exact sequence $\mathbb{Z} \longrightarrow cl\ Q \longrightarrow cl\ (Q - Y) \longrightarrow 0$ we obtain that the sequence:

$\mathbb{Z} \longrightarrow cl\ Q \longrightarrow 0$

$1 \longmapsto Y$

is exact and that means $\mathbb{Z} \longrightarrow cl\ Q$ is injective, therefore,

$cl\ Q \cong \mathbb{Z}$.

The rest of the lemma can be proved by taking a similar sequence.

The scheme R

We define a scheme R and determine some intersection products in the Chowring $A_*(R)$.

Let $V = V^+(y_1y_2 + y_3y_4) \subseteq \mathbb{P}^3(y_1 : \cdots : y_4)$

be a projective variety in $\mathbb{P}^3$.

Let $Q_s := V^+(y_1y_2 + y_3y_4) \subseteq \mathbb{P}^4(y_1 : \cdots : y_5)$ be the projective closure of the affine cone $C(V) \subseteq \mathbb{A}^4(z_1, z_2, \cdots, z_4)$; where $z_i = y_i/y_5$

and:

$P = (0 : 0 : \cdots : 0 : 1) \in \mathbb{P}^4$

be the vertex of this cone i.e. $Q_s = \overline{C(V)} \subseteq \mathbb{P}^4$,

then the projection:

$\pi : Q_s - \{P\} \longrightarrow V$

Induces an isomorphism:

$\pi^* : cl\ V \xrightarrow{\sim} cl\ (Q_s - \{P\}) \cdots\cdots\ (1)$

But it is clear that:

$$Q_s - \{P\} \xhookrightarrow{j} Q_s \,,$$

Therefore, j induces an isomorphism:

$$j^*: cl\ Q_s \xrightarrow{\sim} cl\ (Q_s - \{P\}) \ \cdots\cdots \quad (2)$$

(1) and (2) give $cl\ V \cong cl\ Q_s$.

The generator system of $cl\ Q_s = A_2(Q_s)$ (Chowring) can be found through a generator system of $cl\ V$. Now we blow-up Q_s in $P = (0:\cdots:0:1) \in \mathbb{P}^4$, and obtain the scheme:

$$R := \begin{cases} u_1u_2 + u_3u_4 = 0 \\ \quad x_iu_j = x_ju_i \end{cases}$$

in $\mathbb{A}^4 \times \mathbb{P}^3(x_1, \cdots, x_4, u_1 : \cdots : u_4)$.

The exceptional locus of this blowing up ($R \xrightarrow{f} Q_s$) is:

$K = f^{-1}(P) \cong \mathbb{P}^1 \times \mathbb{P}^1$

The isomorphic types of the components: Alwadi and Dgheim (1988)

(i) By blowing up the A_1 singularity, in the origin of the , we obtain the exceptional locus affine space $\mathbb{A}^5(x_1, \cdots, x_5)$ and:

$$E = f^{-1}(P),$$

$E \cong Q := v^+(y_1^2 + y_2y_3 + y_4y_5) \subseteq \mathbb{P}^4$

In this case E consists of one component only and:

$cl\ Q = Pic\ Q \cong \mathbb{Z}$ $\ (Pic\ Q = \text{Picard group})$

(ii) The exceptional locus E of the A_2 singularity consists of two components $E = E_1 + E_2$, where, $E_1 \cong R$, and $E_2 \cong \mathbb{P}^3$

We know that $A_2 = spec(K[x_1, \cdots, x_5]/\langle f_2 \rangle)$, and

$f_2 = x_1^3 + x_2x_3 + x_4x_5$

(ii) The A_3 -singularity is given in $\mathbb{A}^5$ by

$A_3 = spec(K[x_1, \cdots, x_5]/\langle f_3 \rangle)$, where;

$f_3 = x_1^4 + x_2x_3 + x_4x_5$.

In this case E also consists of

E_1, and E_2, where, $E_1 = R$, and $E_2 \cong Q$.

(iv) In the general case,

$A_n = spec(K[x_1, \cdots, x_5]/\langle f_n \rangle)$, where:

$f_n = x_1^n + x_2x_3 + x_4x_5$,

If and

If

and

Note that, $m = \left[\frac{n+1}{2}\right]$.

CRITERION KLEIMAN (1966)

Let X be a nonsingular complete scheme and let $A_1 = A_1(X)$ be all curves in X, where;
the Chowring Fulton (2008). We assume numerical equivalence). For an we say if and only if

$L \in Pic\ X$ $L \asymp 0$
$L.C = 0, \quad \forall c \in A_1(X)$

$A_1(X)$ denotes $Pic\ X/\asymp$ ("$\asymp$"

numerical equivalence). For an invertible sheaf

If $L \in Pic\ R$ and $L \asymp 0$, then $L \cong O_R$
$L = O_R(a_1H_1 + a_2H_2 + bK)$, where O_R
rings on R.
$L \asymp 0 \Leftrightarrow (L.C) = 0 \quad C$
R

We define two real vector spaces:
$N^1 = N^1(X) = (Pic\ X/\asymp) \otimes_Z R$
$N_1 = N_1(X) = (A_1(X)/\asymp) \otimes_Z R$
We have a perfect pairing of N^1 with N_1 which will be induced by the intersection product:
$N^1(X) \times N_1(X) \longrightarrow R$
$(L,C) \longmapsto (L.C) \coloneqq \deg_C(L \otimes O_C)$
We define also $\overline{NE(X)}$ to be a closed convex cone in
$N_1(X)$, which can be induced by irreducible curves. With the above notations, the Kleiman's criterion take the following form:
"An invertible sheaf $L \in Pic\ X$ is ample if and only if the mapping:
$L: \overline{NE(X)} - \{0\} \longrightarrow R$
$C \longmapsto L.C$ is positive "

THE AMPLE SHEAVES ON THE COMPONENTS WHICH ARE ISOMORPHIC TO *R*

the exceptional locus $E = E_1 + \cdots + E_m$:

$n \equiv 1\ mod\ 2$, then $E_i \cong R$, $(i = 1, \cdots, m-1)$
$E_m \cong Q$.

$n \equiv 0\ mod\ 2$, then $E_i \cong R$, $(i = 1, \cdots, m-1)$

$E_m \cong \mathbb{P}^3$.

It is known that $E_i \cong R$
dimensional schemes. On the other hand, we have also:

$Pic\ R = Z.H_1 \oplus Z.H_2 \oplus Z.K$

(H_1, H_2, and K
(2003))

Lemma

$Pic\ R \cong Pic\ R/\asymp$

Proof:

curve in . In particular this holds for the following curves:
$C = C_{K1},\ C_{K2},\ C_{12}(C_{K1} = k.H_1,\ C_{K2} = k.H_2,\ C_{12} = H_1.H_2)$
In $A_k(R)$ (the intersections in the Chowring), we have:
$(L.C_{K1}) = a_2 - b = 0$
$(L.C_{K2}) = a_1 - b = 0$
$(L.C_{12}) = b = 0$
Therefore, $a_1 = a_2 = b = 0$ thus, $L \cong O_R$.

Proposition

$\overline{NE(R)} = \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12}$,
where C_{K1}, C_{K2}, C_{12}

are as in the last lemma.

Proof:

$N_1(R)$ is a 3-dimensional real vector space,

and $C_{K1}, C_{K2}, C_{12} \in N_1(R)$. If C_{K1}, C_{K2}, C_{12}

are linearly independent, then:

$N_1(R) = \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12}$

But if:

$C = a_1 C_{K1} + a_2 C_{K2} + b C_{12} = 0,$

Then:

$0 = (H_1.C) = a_2 = 0$

$0 = (H_2.C) = a_1 = 0$

$0 - (K.C) - -u_1 - u_2 + b - 0 \Rightarrow b - 0 \Rightarrow u_1 - u_2 - b - 0$

That is, $N_1(R) = \mathbb{R}C_{K1} + \mathbb{R}C_{K2} + \mathbb{R}C_{12}$.

Now $NE(X)$ is the closed cone in , $N_1(R)$

which induced by irreducible curves, that is,

$\overline{NE(R)} = \{\sum r_c.c \bmod \equiv : r_c \geq 0, \text{and all } r_c = 0 \text{ except of finite number of } r_c\}$

We note that c is irreducible over R. We show at this stage: $\overline{NE(R)} = \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12}$,

we first have to show:

$\overline{NE(R)} \subseteq \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12}$ i.e.

we have to show if:

$\sum r_c . c \in \overline{NE(R)},$

Then:

$\sum r_c . c \in \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12}$

Therefore, we have to show if $c \in R$ is an irreducible curve, then:

$c \in \mathbb{R}_+ C_{K1} + \mathbb{R}_+ C_{K2} + \mathbb{R}_+ C_{12},$

Therefore, it is sufficient to show if:

$C \equiv a_1 C_{K1} + a_2 C_{K2} + b C_{12}\ (a_1, a_2, b \in \mathbb{R}),$

then $a_1, a_2, b \geq 0$. For this goal we have the following cases:

$C \subseteq H_1,\ \ C \subseteq H_2,\ \ C \subseteq K$ but in this case:

$(C.H_1) = a_2 \geq 0$

$(C.H_2) = a_1 \geq 0$, and

$(C.K) = -a_1 - a_2 + b \geq 0 \Longrightarrow b \geq a_1 + a_2 \Longrightarrow b \geq 0$

$C \subseteq H_1 \Longrightarrow C = \alpha . C_{K1} + \beta . C_{K2}\ ;\ \ \alpha, \beta \in \mathbb{Z}$

If $C \subseteq C_{K1}$ (this corresponds

$C \subseteq C_{K1},\ C = 1.C_{K1} + 0.C_{12}$) clear.

If $C \subseteq C_{12}$ (corresponds) $C = C_{12}$

If $C \subseteq C_{K1}$ and $C \subseteq C_{12}$ that is, $(C.C_{K1}) \geq 0$ and

$(C.C_{12}) \geq 0$ but on H_1 we have

$C_{K1}^2 = -1, C_{K1}.C_{12} = \mathbf{1}$ and $C_{12}^2 = 0$

Alexiou and Alwadi (2003), therefore:

$$\alpha,\beta \geq 0 \Leftarrow \begin{cases}(C.C_{K1}) = -\alpha+\beta \geq 0 \\ (C.C_{12}) = \alpha \geq 0\end{cases}$$

$C \subseteq H_2$ (symmetric to(2)).

$C \subseteq K,\ K \cong \mathbb{P}^1 \times \mathbb{P}^1 \qquad \Longrightarrow$

Lazarsfeld (2005)

$C = \alpha_1.C_{K1} + \beta_1.C_{K2}$ in $(\mathbb{P}^1 \times \mathbb{P}^1) = Z.C_{K1} \oplus Z.C_{K2}$

If $C \subseteq C_{K1}$ $(C = 1.C_{K1} + 0.C_{K2})$.

If $C \subseteq C_{K2}$ and $C \subseteq C_{K2}$ i.e. $(C.C_{k1}) \geq 0$ and

$(C.C_{K2}) \geq 0$ but on $C_{K1}.C_{K2} = 1, C_{K1}^2 = C_{K2}^2 = 0$

$(C.C_{K1}) = \beta_1 \geq 0, \qquad (C.C_{K2}) = \alpha_1 \geq 0$

Finally, we deduce the following:

Theorem

$L = O_R(a_1H_1 + a_2H_2 + bK)$ is ample over $R \Leftrightarrow a_1 - b, a_2 - b, b > 0$

Proof:

$(\Longrightarrow)$

Let L be ample, then by the Kleiman's criterion we have:

$(L.C_{K1}) > 0, \qquad (L.C_{K2}) > 0, \qquad$ and $(L.C_{12}) > 0,$

But:

$L.C_{K1} = L.K.H_1 = a_2 - b > 0$

$L.C_{K2} = L.K.H_2 = a_1 - b > 0$

$L.C_{12} = L.H_1.H_2 = b > 0$

$(\Longleftarrow)$

Can be obtained from the above proposition.

DISCUSSION AND CONCLUSION

This paper put conditions for ampleness on the irreducible components of the canonical resolution of the simple 4-dimensional singularity A_n. It will be fruitful to be done as a similar study in higher dimensions, because for 3-dimensional it was done Roczen (1984), also it will be fruitful to be done as a similar study for another simple singularities (D_n , E_n). Our studies enable us to study the so called "negative embedded" of the exceptional locus.

The conditions that we found are important cohomological properties of the irreducible components of the canonical resolution of the exceptional locus.

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