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DEVELOPMENT OF A TRANSMUTED LOMAX GAMMA DISTRIBUTION AND APPLICATION

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Abstract

New families of continuous probability distributions have been introduced; the so-called Development of Transmuted Lomax Gamma Distribution application and its properties were proposed and studied. Various structural properties, including explicit expressions for the moments, quantile functions, order statistics, survival functions, hazard functions, and estimations of new distributions were derived. The performances of the maximum -likelihood estimates of the parameters of the Transmuted Lomax Gamma family were evaluated through a simulation study. After applying the new distribution to real data, we compared its performance to that of other competing distributions and found that the Transmuted Lomax Gamma distribution performed better when using BIC, AIC, and CAIC. Furthermore, we also concluded that the distribution can be used to model highly skewed data (skewed to the right).

INTRODUCTION

The Transmuted Lomax-Gamma Distribution is a versatile statistical distribution that has been applied in various fields, including engineering, economics, and environmental studies. This distribution is a compound distribution that arises from a mixture of Lomax and Gamma distributions, offering flexibility in capturing various shapes and tail behaviors. To provide a brief introduction, the Lomax distribution, also known as the Pareto Type II distribution, is a heavy-tailed distribution commonly used to model extreme value phenomena. Transmuted Lomax-Gamma Distribution: Theory and Applications. Communications in Statistics - Theory and Methods.

In contrast the Gamma distribution is a flexible continuous probability distribution that is often employed in various areas, such as reliability analysis, queuing theory, and income modeling. The transmuted Lomax-Gamma distribution results from the combination of these two distributions, to forming a new distribution that inherits properties from both parent distributions. This composite allows for a rich representation of data with variability, skewness, and heavy-tailed characteristics, making it a useful tool for modeling real-life phenomena. The transmuted Lomax-Gamma distribution is defined by a set of parameters that govern its shape, scale, and transmutation features. These parameters enable practitioners to tailor the distribution to the specific features of the data at hand, thereby enhancing its applicability to diverse problems. One of the key advantages of the

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transmuted Lomax-Gamma distribution is its ability to capture diverse shapes of data, including unimodal, bimodal, and skewed distributions. This flexibility is particularly valuable when the underlying data exhibit complex and varied patterns.

Furthermore, the transmuted Lomax-Gamma distribution provides a rich framework for statistical inference, including parameter estimation, hypothesis testing, and model assessment. This ensures that practitioners can make informed decisions based on rigorous data analyses. The study of the transmuted Lomax-Gamma distribution was motivated by need for a flexible and versatile statistical model that combines the characteristics of the Lomax and Gamma distributions. This distribution has gained attention recently due to its applicability in various fields, such as reliability analysis, survival modeling, and risk assessment. The transmuted Lomax-Gamma distribution is a flexible and customizable statistical model that combines the characteristics of Lomax and Gamma distributions through a transmutation process. This distribution provides researchers with a versatile tool for modeling several data patterns with varying skewness and tail characteristics. By adjusting the transmutation parameter, the shape and behavior of the distribution can be tailored to suit different types of data, making it a valuable asset for statistical analysis.

In this paper, article we present the cumulative distribution function (cdf) and the probability density function (pdf) of the Transmuted Lomax Gamma family of distributions, using the Lomax Gamma family proposed by Cordeiro et al (2019). They defined the cumulative distribution function (cdf) and the probability density function of the Lomax Gamma family of distribution as; if G(x) denotes the cumulative distribution function (cdf) of a random variable, the Lomax Gamma cdf family is

$$F_{LG}(x) = 1 - \alpha^{\beta} + \left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta} \quad (1)$$

$$CDF = \begin{cases} 0 & x < 0 \\ \dfrac{1}{\lambda^{\alpha}\Gamma(\alpha)} \displaystyle\int_0^x x^{\alpha-1} e^{-\frac{x}{\lambda}} dx & x \geq 0 \end{cases}$$

Its associated pdf is:

$$f_{LG}(x) = \frac{\beta\alpha^{\beta} g(x)}{\left(1 - G(x)\right)^2}\left(\alpha + \frac{G(x)}{1 - G(x)}\right)^{-\beta+1} \quad (2)$$

Where $\alpha, \beta > 0$ *are the shape parameters*

The Transmuted family of distributions was developed by Shaw and Buckley (2009); they were prompted by the need to provide parametric families of distribution that would be flexible and, at the same time, would be expected to be useful not only in finance but also in other wider areas in Statistics.

The formulation of the transmuted family of distributions involved the use of a transmutation map, which was described by Shaw and Buckley (2007) as a function comprises the cdf of one distribution with the quantile function of another. The approach was aimed at inducing skewness or kurtosis, as may occur in available distributions. Therefore, Shaw and Buckley (2009) defined the transmuted gamma family of distributions as follows:

$$F_{TLG} = \frac{\beta\alpha^{\beta} g(x)}{G(x)^2}\left(\alpha + \frac{G(x)}{\breve{G}(x)}\right)^{-\beta-1} \left\{(1 + \lambda) - 2\lambda\left\{1 - \alpha^{\beta}\left(\alpha + \frac{G(x)}{\breve{G}(x)}\right)^{-\beta}\right\}\right\} (3)$$

$$f_{TLG} = (g(x;\, J) = 1 \; (4)$$

Where $g(x)$ and $G(x)$ are the baseline cdf and pdf respectively for $|\lambda| \leq 1$

Now, they cdf and pdf of new Transmuted Lomax-G family of distributions are defined by combining the two cumulative distribution function of equation (1) and (3), as well as the two-probability density function of equation (2) and (4), respectively. This yields the cdf and pdf of the Transmuted Lomax Gamma family of distributions, as given below:

$$F_{TLG}(x;\alpha,\beta,\lambda,\theta)=\left\{1-\alpha^{\beta}\left(\alpha+\frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right\}(1+\lambda)-\lambda\left\{1-\alpha^{\beta}\left(\alpha+\frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right\}^{2} \quad (7)$$

$$f_{TLG}(x;\alpha,\beta,\lambda,\theta)=\frac{\beta\alpha^{\beta}g(x)}{\bar{G}(x)^{2}}\left(\alpha+\frac{G(x)}{\bar{G}(x)}\right)^{-\beta+1}\left\{(1+\lambda)-2\lambda\left[1-\alpha^{\beta}\left(\alpha+\frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right]\right\} \quad (8)$$

Where $\bar{G}(x)=\bar{G}(x)=1-G(x)$ and θ is a vector of parameters for the baseline distribution, $\alpha,\beta>0$ are the shape parameters, $|\lambda|\leq 1$ is the transmuted parameter. For $\lambda=0$, we have the Lomax Gamma proposed by Cordeiro et al. (2019). Henceforth, we denote random variable X having PDF (6) as follows: $X\sim TLG(x;\alpha,\beta,\lambda,\theta)$.

1 Transmuted Lomax Gamma Distribution Validation

If X is a continuous probability density function, the following condition must be satisfied:

1. $f(x)\geq 0$ (non-negativity property)
2. $\int_{-\infty}^{\infty} f(x)dx=1$
3. $P(a\leq x\leq b)=\int_{a}^{b} f(x)dx$

The properties of the Model

i $\Gamma(\alpha)=\int_{0}^{\infty} x^{\alpha-1}e^{-x}dx;$

ii $\int_{0}^{\infty} x^{\alpha-1}e^{-\lambda x}dx=\frac{\Gamma(\alpha)}{\lambda^{\alpha}}, for\ \lambda>0;$

iii $\Gamma(\alpha+1)=\alpha\Gamma(\alpha);$

iv $\Gamma(n)=(n-1)!, for\ n=1,2,3,\ldots;$

v $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$

Proof: Consider the new pdf of the Transmuted Lomax Gamma family of distributions in (8).

$$f_{TLG}(x;\alpha,\beta,\lambda,\theta)=\frac{\beta\alpha^{\beta}g(x)}{\bar{G}(x)^{2}}\left[\alpha+\frac{G(x)}{\bar{G}(x)}\right]^{-\beta+1}\left\{(1+\lambda)-2\lambda\left[1-\alpha^{\beta}\left(\alpha+\frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right]\right\} \quad (9)$$

$for\ \alpha,\beta,\lambda,\theta>0$

For (1) to be valid, $\int_{-\infty}^{\infty} f(x)dx=1$ (10)

$$F_{TLG}(x,\alpha,\beta,\lambda,\theta) = \frac{\beta\alpha^{\beta}(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)} \times (1 - G(X))^2\left(\alpha + \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)} \times (1 - G(X)\right)^{-\beta+1}\left\{(1+\lambda) - 2\lambda\left[1 - \alpha^{\beta}\left(\alpha + \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)} \times (1 - G(X))\right)^{-\beta}\right]\right\}$$

$$F_{TLG}(x,\alpha,\beta,\lambda,\theta) = \frac{\beta\alpha^{\beta}(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)} \times \left(1-\frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)}\right)^{2}\left[\alpha + \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)} \times \left(1 - \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)}\right)\right]^{-\beta+1}\left\{(1+\lambda) - 2\lambda\left[1 - \alpha^{\beta}\left(\alpha + \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)} \times 1 - \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)}\right)^{-\beta}\right]\right\}$$

If we expand the above, something can be eliminated, and then we have that in the following: $F_{TLG}(x,\alpha,\beta,\lambda,\theta) = \frac{\beta\alpha^{\beta}(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}[\alpha+1]^{-\beta+1}\{(1+\lambda)-2\lambda[1-\alpha^{\beta}(\alpha+1)^{-\beta}]\}$

$\boldsymbol{set, \beta = 1}$

$$\frac{1\alpha^{1}(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}[\alpha+1]^{-1+1}\{(1+\lambda)-2\lambda[1-\alpha^{1}(\alpha+1)^{-1}]\}$$

$$\frac{\alpha(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}[1]^{0}\left\{(1+\lambda)-2\lambda\left[1-\alpha\left[\frac{1}{(1)}\right]\right]\right\}$$

By applying the integration, we obtain the folllowing.

$$= \int_{0}^{1}\frac{\lambda(2\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}$$

$$\frac{2\lambda\,\lambda^{\alpha}}{\Gamma(\alpha)}\int_{0}^{1}x^{\alpha-1}e^{-\lambda x}d\lambda$$

Recall that from property (2) of the gamma function:

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha)}{\lambda^{\alpha},}\int_{0}^{1}2\lambda d\lambda$$

$$= \int_0^1 2\lambda d\lambda$$

$$\alpha, \beta, \lambda, \theta > 0$$

2 Quantile Function

The Quantile function is used to partition probability distributions, obtain the median of a distribution, and simulate random numbers.

Let $F(x) = u\ such\ that\ Q(u)$

$F\big(Q(u)\big) = u \Rightarrow Q(u) = F^{-1}(u) for 0 < u < 1$

We can convert Eq. (7) to obtain the quartile function of the TLG family as follows:

$$u = 1 - \alpha^{\beta}\left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta}(1+\lambda) - \lambda\left\{1 - \alpha^{\beta}\left(\alpha^{\beta} + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right\}^{2} \quad (12)$$

where $U \sim U(0.1)$, i.e. uniformly distributed with intervals 0 and 1,

$$F_{TLG}(x, \alpha, \beta, \lambda, \theta) = \frac{\beta\alpha^{\beta}(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)} \times \left(1 - \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)}\right)^{2}\left[\alpha + \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)} \times \left(1 - \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)}\right)\right]^{-\beta+1}\left\{(1+\lambda) - \lambda\left[1 - \alpha^{\beta}\left(\alpha + \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)} \times 1 - \frac{x^{\alpha-1}e^{\frac{-x}{\lambda}}}{\lambda^{\alpha}\Gamma(\alpha)}\right)^{-\beta}\right]\right\}$$

If we expand the above, something can be eliminated; thus, we obtain the following.

$$F_{TLG}(x,\alpha,\beta,\lambda,\theta) = \frac{\beta\alpha^{\beta}(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}[\alpha + 1]^{-\beta+1}\{(1+\lambda) - \lambda[1-\alpha^{\beta}(\alpha+1)^{-\beta}]\}$$

$$\boldsymbol{set, \beta = 1}$$

$$\frac{1\alpha^{1}(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}[\alpha+1]^{-1+1}\{(1+\lambda) - \lambda[1-\alpha^{1}(\alpha+1)^{-1}]\}$$

$$\frac{\alpha(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}[\alpha+1]^{0}\{(1+\lambda) - \lambda[1-\alpha(\alpha+1)^{-1}]\}$$

$$\frac{\alpha(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}[\alpha+1]^{0}\left\{(1+\lambda) - \lambda\left[1-\alpha\left[\frac{1}{\alpha+1}\right]\right]\right\}$$

By applying the integration, we obtain the following.

$$= \int_0^1 \frac{\lambda(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}$$

$$\frac{\lambda\,\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^1 x^{\alpha-1}e^{-\lambda x}d\lambda$$

Recall that from property (2) of the gamma function:

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha)}{\lambda^{\alpha},}\int_0^1 \lambda d\lambda$$

$$= \int_0^1 \lambda d\lambda$$

Where $G(x)^{-1}$ is the quartile function baseline distribution $\lambda \neq u$, and $\alpha, \beta > 0$

3 Moment Generating Function (mgf)

The moment generating function $M_x(t)$ of a random variable X is expressed as follows:

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Therefore, the mg of the TLG family can be expressed as follows:

$$M(t) = \sum_{p=0}^{\infty}\frac{t^p}{p!}\int_{-\infty}^{\infty} x^p\, f(x)dx$$

$$= \sum_{-\infty}^{\infty}\frac{t^p}{p!}\mu_r^1 \ (1)$$

γ is Gamma with parameters α and λ then, I should find the mean, variance, skewness and kurtosis.

Mean

TLG: Transmuted Lomax Gamma

If X~Gamma(α, λ), then

$E(X) = \frac{\alpha}{\lambda}, Var(X) = \frac{\alpha}{\lambda^2}$.

$$E(X) = \int_0^{\infty} Xf_X(x)dx = \int_0^{\infty} x \times \frac{\lambda^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}dx \ \ (15)$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x \times x^{\alpha - 1} e^{-\lambda x} dx$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha} e^{-\lambda x} dx$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha + 1)}{\lambda^{\alpha+1}}$$

(Using property 2 of the gamma function)

$$= \frac{\alpha\Gamma(\alpha)}{\lambda\Gamma(\alpha)}$$

(Using property 3 of the gamma function)

$$E(X) = \frac{\alpha}{\lambda}.$$

Variance

Similarly, we obtain $E(X^2)$:

$E(X^2) = \int_0^{\infty} x^2 f_X(x) dx$ (16)

$$= \int_0^{\infty} x^2 \times \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\lambda x} dx$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^2 \times x^{\alpha - 1} e^{-\lambda x} dx$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha+1} e^{-\lambda x} dx$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(\alpha + 2)}{\lambda^{\alpha+2}}$$

(Using property 2 of the gamma function)

$$= \frac{(\alpha + 1)\Gamma(\alpha + 1)}{\lambda^2\Gamma(\alpha)}$$

(Using property 3 of the gamma function)

$$= \frac{(\alpha + 1)\alpha\Gamma(\alpha)}{\lambda^2\Gamma(\alpha)}$$

(Using property 3 of the gamma function)

$$= \frac{(\alpha + 1)\alpha}{\lambda^2}$$

So, we conclude that

$$var(x) = E(X) - E(X)^2$$

$$var(x) = \frac{(\alpha + 1)\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2}$$

$$Var(X) = \frac{\alpha}{\lambda^2}.$$

Skewness

$$\frac{\mathbf{2}}{\sqrt{\sigma}}$$

$\frac{\mu_3}{\sigma^3} = \frac{\alpha\lambda^3(\alpha+1)(\alpha+2)}{\alpha^{3/2}\lambda^3}$ (18)

$= \frac{1}{\sqrt{\alpha}}(\alpha + 1)(\alpha + 2)$ (19)

$= 2\alpha\lambda^3$

Kurtosis

Let M_4 ---> G_4

$M_4 = \frac{n(n+1) \sum(xi\, x)^4 - 3(n-1)\left(\sum(xi - x)^2\right)^2}{(n-1)(n-2)(n-3)}$ (20)

Kurtosis () is a measure of the flatness or peakness of a distribution.

The fourth moment is estimated as the fourth moment divided by the standard deviation to the power of 4:

$$\alpha_4 = \frac{G_4}{G_\alpha^4}$$

4 Renyi and Entropy

$f_{TLG}(x; \alpha, \beta, \lambda, \theta)^r = \frac{(\beta\alpha^\beta)^r g(x)^r \left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{r(\beta+1)}}{\bar{G}(x)^2} \left\{ r(1+\lambda)^r - (2\lambda)^r \left[1 - \alpha^\beta \left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-r\beta}\right]\right\}$ (21)

By applying a power expansion similar to that in the linear presentation section

$$I_r(x) = (1-r)^{-1} \log\left\{\int_0^\infty \left[\sum_{i,j=0}^{\infty} \left(\frac{\beta}{\alpha}\right)^r \alpha^{-i} \frac{\sqrt{i+2r+j}}{j!\sqrt{1+2r}} G(x;\theta)^{i+j} g(x;\theta)^r \left[\binom{-r(\beta+1)}{i}(1+\lambda)^r - (2\lambda)^r \binom{-r(\beta+1)}{i} + (2\lambda)^r \binom{-r(2\beta+1)}{i}\right]\right]\right\}$$

$$I_r(x) = (1-r)^{-1} \log\left\{\int_0^\infty \sum_{i,j=0}^{\infty} B_{ij} E_{i+j}(x)\, dx\right\} \quad (22)$$

Where;

$$B_{ij} = \left(\frac{\beta}{\alpha}\right)^r \alpha^{-i} \frac{\sqrt{i+2r+j}}{j!\sqrt{i+2r}} \left[\binom{-r(\beta+1)}{i}(1+\lambda)^r - (2\lambda)^r \binom{-r(\beta+1)}{i} + (2\lambda)^r \binom{-r(2\beta+1)}{i}\right]$$

Furthermore,

$E_{i+j}(x) = g(x)^r, G(x)^{i+j}$

5 Order Statistics

The pdf for the r[th] order statistics$X_{r=n}$, of a random sample $x_1, x_2, x_3, \ldots x_n$ of sets n$f_{r=n}(x)$ is determined as

$$f_{r=n}(x) = \frac{n!}{(r-1)!\,(n-r)!} F_{TLG}(x)^{r-1}(1 - F_{TLG}(x)^{n-r}) F_{TLG}(x), for\ r = 1,2,3, \ldots n \ (23)$$

$F_{min}(x) = F_1(x) = 1 - p\,(X_{min} > X)$

$= 1 - p\,(X_1 > X, X_2 > X, \ldots X_n > X)$

$= 1 - (1 - F_1(x))(1 - F_2(x)) \ldots (1 - F_n(x))$

$= 1 - p\,(X_1 > X, X_2 > X, \ldots X_n > X)$

$= 1 - (1 - F(x))^n$

$F_{min}(x) = \frac{d}{dx}(1 - F(x))^n = n\,(1 - F(x))^{n-1} F(x)$

The first-order statistic is the smallest sample value (i.e the minimum) once the values have been placed in order.

By expanding $[F_{TLG}(x)]^{r-1}$ using binomial series expansion given

$$[F_{TLG}(x)]^{r-1} = \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \left(1 - F_{TLG}(x)\right)^k$$

Substituting back into equation (12), we obtain

$$f_{r=n}(x) = \frac{n!\, f(x)}{(r-1)!\,(n-r)!} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \left(1 - F_{TLG}(x)\right)^{n-r-k}$$

$$= f_{TLG}(x)[1 - F_{TLG}(x)]^{n-r+k}$$

$$= \frac{\beta\alpha^{\beta} g(x)}{\bar{G}(x)^2} \left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta-1} \left\{(1-\lambda) - 2\lambda \left[1 - \alpha^{\beta}\left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right]\right\} \left[\alpha^{\beta}\left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right]^{n-r+k}$$

$$= \frac{\beta g(x)\alpha^{\beta+n-r+k}}{\bar{G}(x)^2} \left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{n+k-r-\beta-1} (1-\lambda) \frac{2\lambda\beta\alpha^{\beta+n-r+k} g(x)}{\bar{G}(x)^2} \left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{n-r+k-\beta-1}$$
$$+ \frac{2\lambda g(x)}{\bar{G}(x)^2} \beta\alpha^{n-2\beta-r+k-1} \left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{n-2\beta-r+k-1}$$

$$= \sum_{ij=0}^{\infty} g(x)G(x)^{i+j} \beta \frac{\sqrt{i+j+2}}{j!\sqrt{i+j}} \left[\binom{n+k-r-\beta-1}{i} \alpha^{2n-2r+2k-1-i}(1+\lambda)\right.$$
$$\left. - 2\lambda \binom{n+k-r-\beta-1}{i} \alpha^{2n-2r+2k+1} + 2\lambda \binom{n-2\beta-r+k-1}{i} \alpha^{2n-2\beta-2r+2k-\lambda-1}\right]$$

Therefore,

$$f_{r=n}(x) = \frac{n! \sum_{0}^{r=1} (-1)^k \binom{r-1}{k}}{(r-1)!\,(n-r)!} \sum_{i,j=0}^{\infty} O_{ij}\, g(x)G(x)^{i+j} \quad (24)$$

Where;

$$O_{ij} = \beta \frac{\sqrt{i+j+2}}{j!\sqrt{i+j}} \left[\binom{n+k-r-\beta-1}{i} \alpha^{2n-2r+2k-1-i}(1+\lambda) - 2\lambda \binom{n+k-r-\beta-1}{i} \alpha^{2n-2r+2k+1}\right.$$
$$\left. + 2\lambda\alpha^{2n-2\beta-2r+2k-2i} \binom{n-2\beta-r+k-1}{i}\right]$$

6 Hazard Function

The hazard function is the probability that a component will die over a certain period.
The hazard function is defined as follows:

$$h(x) = \frac{f(x)}{1-F(x)} \quad (25)$$

Here, f(x) and F(X) are the pdf and cdf of the Transmuted Lomax-G family of distributions given in equations (7) and (8), respectively.

Substituting for f(x) and F(X) and simplifying gives

$$h_{TLG}(x; \alpha, \beta, \lambda, \theta) = \frac{f(x)}{1-F(x)}$$

$$= \frac{\beta g(x;\theta) \left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta+1} \left[1 + \lambda - 2\lambda\left(1 - \alpha^{\beta}\left(\alpha^{\beta} + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right)\right]}{[1 - G(x)]^2} \quad (26)$$

$$\frac{2\alpha\lambda x^{\alpha-1}\,[1-x^{2\alpha}]^{\lambda-1}}{[1-x^{2\alpha}]^{\lambda}}$$
$$\frac{2\alpha\lambda\,[1-x^{2\alpha}]^{\lambda-1}}{[1-x^{2\alpha}]^{\lambda}\,[1-x^{2\alpha}]}$$
$$\frac{[2\alpha\lambda x^{\alpha-1}\,]}{1-x^{2\alpha}}$$
$$\frac{2\alpha\lambda x^{\alpha-1}}{1-x^{2\alpha}}$$

7 Survival Function

The Survival function is the probability that an individual or a system will not fail for a given time. Mathematically, it is given by given as

$$S(x) = 1 - G(x)$$

Here, F(x) is the cdf of the new Transmuted Lomax-G family of distributions.

By substituting, we have:

$$S_{TLG}(x;\alpha,\beta,\lambda,\theta) = 1 - G(x) = \alpha^{\beta}\left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta} \quad (27)$$

$$S(x) \;=\; 1 -\; G(x)$$
$$1 - \{1 - [1 - x^{2\alpha}]^{\lambda}\}$$
$$\{1 - 1 + [1 - x^{2\alpha}]^{\lambda}\}$$
$$S(x) \;= [1 - x^{2\alpha}]^{\lambda}$$

8 Estimation

In this section, we estimate estimation of the parameters of the Transmuted Lomax-G family of distributions (TLG) using the Maximum Likelihood estimation method. Let $X_1, X_2, X_3, \dots, X_n$ be a random sample of size n from the TLG family of distributions, and the likelihood function is expressed as follows:

$$\prod_{i=1}^{n} f(x,\alpha,\beta,\lambda,\theta) = \prod_{i=1}^{n}\left[\frac{\beta\alpha^{\beta}g(x)}{\bar{G}(x)}\left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta-1}\left\{(1+\lambda) - 2\lambda\left[1 - \alpha^{\beta}\left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right]\right\}\right] \quad (28)$$

Furthermore, the corresponding log-likelihood function is given by

$$l = nlog(\beta) + nlog\left(\alpha^{\beta}\right) + \sum_{i=1}^{n} log\left(g(x)\right) - \sum_{i=1}^{n} log\left(\bar{G}(x)\right) - (\beta+1)\sum_{i=1}^{n} log\left(\alpha + \frac{G(x)}{\bar{G}(x)}\right) + \sum_{i=0}^{n} \log\left[(1+\lambda) - 2\lambda\left[1 - \alpha^{\beta}\left(\alpha + \frac{G(x)}{\bar{G}(x)}\right)^{-\beta}\right]\right]$$

Then, can be rewritten as follows:

$$l = nlog\beta + \beta nlog\alpha + \sum_{i=1}^{n} logg(x) - \sum_{i=1}^{n} log\bar{G}(x) - (\beta+1)\sum_{i=1}^{n} log(\alpha + Z_i) + \sum_{i=1}^{n} log\left((1+\lambda) - 2\lambda\left[1 - \alpha^{\beta}(\alpha + Z_i)^{-\beta}\right]\right) \quad (29)$$

Where;

$$Z_i = \frac{G(x)}{\bar{G}(x)}$$

By differentiating equation (25) partially with respect to the parameters, we obtain the following:

$U(\psi) = \left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \theta}\right)^T$ Which is the score function as follows

$$\frac{\partial l}{\partial \alpha} = \frac{n\beta}{\alpha} - (\beta+1)\sum_{i=1}^{n}\frac{1}{(\alpha+Z_i)} + \frac{2\lambda\alpha^{\beta}\beta(\alpha+Z_i)^{-\beta}[\alpha^{-1}-(\alpha+Z_i)^{-1}]}{[1+\lambda-2\lambda(1-\alpha^{\beta}(\alpha+Z_i)^{-\beta})]} \quad (30)$$

$$\frac{\partial l}{\partial \lambda} = \frac{-1+2\alpha^{\beta}(\alpha+Z_i)^{-\beta}}{[(1+\lambda)-2\lambda(1-\alpha^{\beta}(\alpha+Z_i)^{-\beta})]} \quad (31)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + nlog\alpha - \sum_{i=1}^{n} log(\alpha+Z_i) + \frac{2\lambda\alpha^{\beta}\text{In}\alpha(\alpha+Z_i)^{-\beta}+(\alpha+Z_i)}{(\alpha+Z_i)^{\beta}[1+\lambda-2\lambda(1-\alpha^{\beta}(\alpha+Z_i)^{-\beta})]} \quad (32)$$

$$\frac{\partial l}{\partial \theta} = \sum_{i=1}^{n}\frac{g(x)}{\bar{G}(x)}\sum_{i=1}^{n}\frac{g(x)}{\bar{G}(x)} - (\beta+1)\sum_{i=1}^{n}\frac{g(x)}{[1-G(x)]^2(\alpha+Z_i)} - \sum_{i=1}^{n}\frac{2\lambda\alpha^{\beta}\beta(\alpha+Z_i)^{-\beta-1}g(x)}{[1-G(x)]^2[1+\lambda-2\lambda(1-\alpha^{\beta}(\alpha+Z_i)^{-\beta})]} \quad (33)$$

Setting the equations above to zero and solving them simultaneously also yields the Maximum likelihood estimations of the four parameters.

Simulation Studies

Consider the equation $F(x) - u = 0$, u is an observation drawn from uniform distribution $(0, 1)$ and $F(x)$ the cdf of the Transmuted Lomax-G family of distributions was used to carry out the simulation study and generate data from Transmuted Lomax Gamma distributions using a Monte Carlo simulation. The Monte Carlo simulation is described by the different sizes of the samples in the quantile function of the transmuted Lomax gamma distributions.

Real Dataset:

There are two datasets for the application of this research work article; the first dataset is the sum of skin folds dataset, which contains 202 observations. These data were the sum of skin folds in 202 athletes from the America Institute of Sports, while the second dataset was the Duncan data. For these data, we fitted and tested the transmuted Lomax Gamma (TLG) distribution defined in (24) and also compared its fit with the following distribution models, "Transmuted Gamma by Adeyinka (2020)

Their corresponding densities are as follows:

1. Transmuted Gamma by Adeyinka (2020)

$$F_{TLG}(x, \alpha, \beta, \lambda, \theta) = \frac{\beta\alpha^{\beta}(\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x})}{\Gamma(\alpha)}[\alpha + 1]^{-\beta+1}\{(1+\lambda) - 2\lambda[1-\alpha^{\beta}(\alpha+1)^{-\beta}]\}$$

2. Probability density function of Gamma, cumulative density function, and quantile function of Gamma
3. Hazard and survival functions of gamma

Simulation Results for TLG Distribution

A Monte Carlo simulation was performed and the results of the mean, bias, and mean squared error of the estimated parameter values are presented in Tables 4.2.1. The Monte Carlo simulation is described below:

i. For the parameter values of Transmuted Lomax Gamma Distribution, samples of different sizes were generated with small parameter values ($\alpha = 0.5, \beta = 1.6, \lambda = 0.8, \theta = 2.9$) using the quantile function.
ii. Using the maximum-likelihood method, we computed the MLE of $\hat{\alpha}_i, \hat{\beta}_i, \hat{\lambda}_i, and\ \hat{\theta}_i$ for the i^{th} replicate.
iii. Steps (i) and (ii) are replicated N = 300 times.
iv. The mean, bias, and mean squared error for each sample size n were computed as follows:

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} \mu_i\,, Bias(\hat{\mu}) = (\hat{\mu} - \mu), MSE(\hat{\mu}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\mu}_i - \mu)^2 \quad (34)$$

Where $\hat{\mu}_i = \left(\hat{\alpha}_i, \hat{\beta}_i, \hat{\lambda}_i, \hat{\theta}_i\right)$ are the MLE for the i^{th} replicate. The same number of dimensions are n = 50, 150, 300, and 400.

Table 4.3: Simulation Results for the TLG Distribution

N	Properties	$\alpha = 0.5$	$\beta = 1.6$	$\lambda = 0.8$	$\theta = 2.8$
50	Mean	0.8671	0.9766	1.9313	2.6947
	Bias	0.3671	-0.6234	1.1313	-0.2053
	MSE	5.3984	5.6935	1836.327	4.0996
150	Mean	0.6246	0.9449	0.5019	1.9534
	Bias	0.1246	-.0.6551	-0.2981	-0.9466
	MSE	0.1294	0.7349	0.2845	2.6235
300	Mean	0.6174	0.9735	0.6252	1.5877
	Bias	0.1174	-0.6265	-0.1748	-1.3129
	MSE	0.0713	0.9538	0.1272	2.7146

The table 4.3 presents and evaluates the behavior of the mean, bias, and mean squared error (MSE) of the estimates, and we can see clearly that the values of the bias and MSEs decrease as the sample size increases, they all approached zero. Also, the estimates tend to the initial value, which shows that the estimates are unbiased and efficient, and the MSEs tending to zero shows precision of the estimates.

9 Application's Results

We illustrate the application of the TLG distribution to two data set; the data set of Sum of skins folds data and Duncan data, as reported in The Generalized Odd Gamma-Family of Distributions: Properties and Applications.

Table 4.4: MLEs for the Sum of Skinfold and Duncan Data

Data sets	**Models**	**MLEs**			
		α	β	λ	θ
Sum of skin fold data	TLGD	2.98125744	0.06399661	7.79332026	0.06632972
	ECTED	3.90751107	0.05450552	6.28645295	1.46156475
	WGED	0.14536807	0.44556179	0.05207786	
	MOLED	2.28932487	4.81178314		
Duncan data	TLGD	4.27891236	0.19560368	9.72772822	0.02785846
	ECTED	6.77114915	0.04800918	7.88137146	4.80175080
	WGED	0.005943488	0.093204440	0.542105630	
	MOLED	4.325145080	5.924578863	0.009126324	

4.2.3 Densities plots

The estimated density plots of the datasets are presented in fig. 1, 2, 3, 4, and 5, respectively. These include plots of the cdf and pdf, hazard rate, and survival rate of the (TLG) distribution. The plots are as follows:

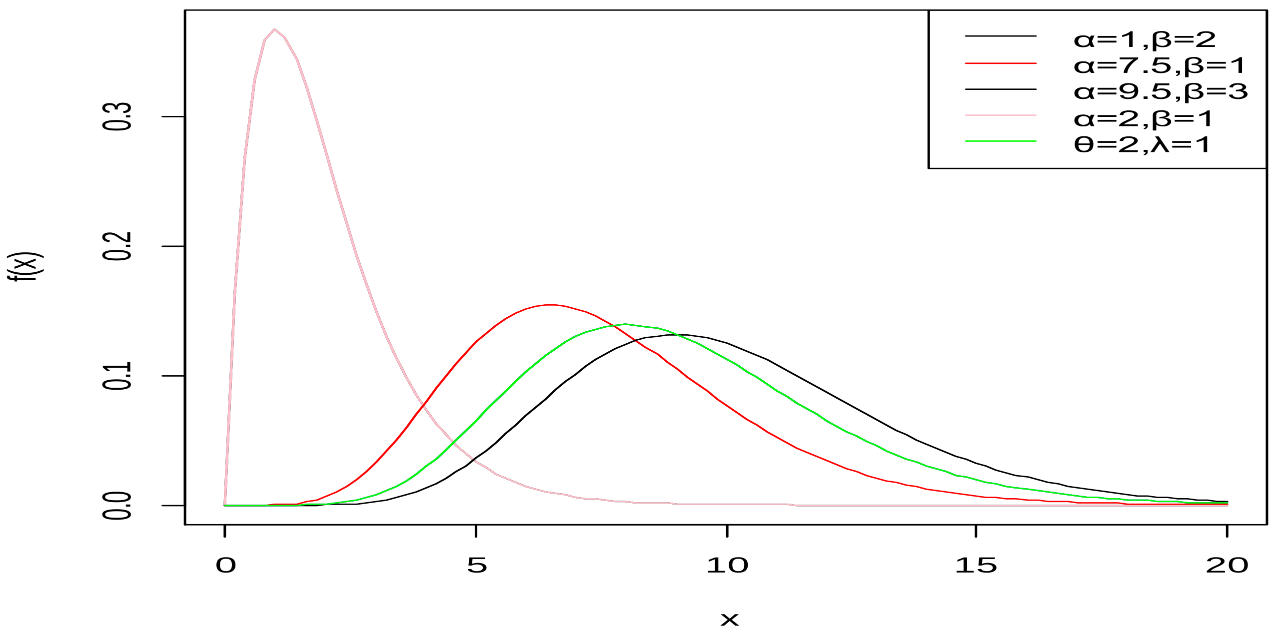

Fig. 4.1: Probability density function (PDF) of the TLG distribution.

PDF of the TLG distribution for selected parameter values $\theta, \lambda, \alpha\ and\ \beta\ with\ \gamma = 1$.

The plot for the PDF reveals that the TLG distribution is positively skewed and thus is a good model for a positively skewed dataset.

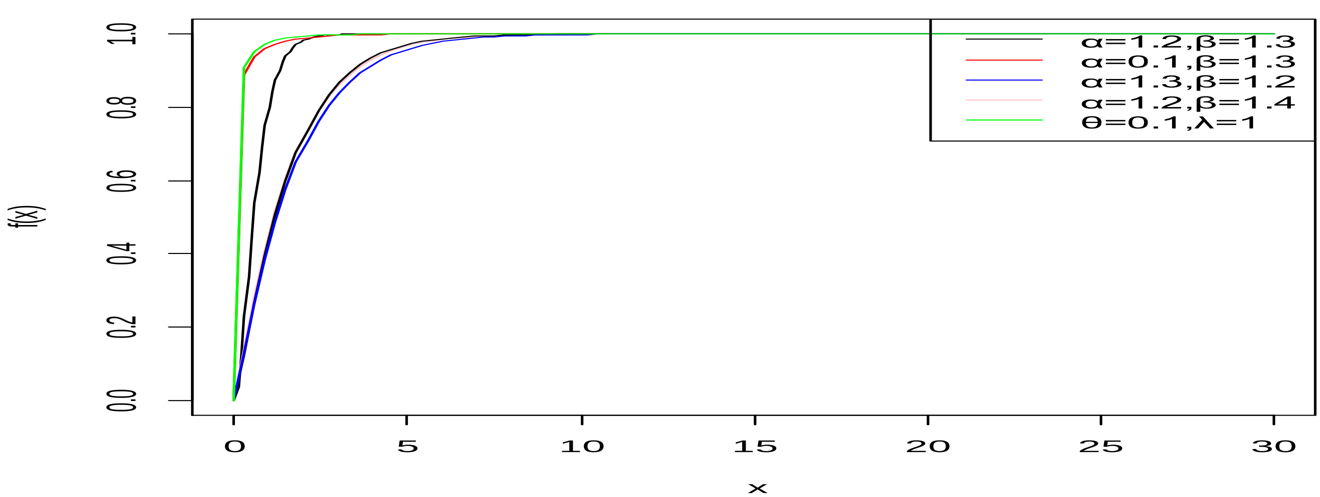

Fig. 4.2: Cumulative density function (CDF) of the TLG distribution.

Figure 4.2 shows some possible shapes of the cumulative distribution function of the TLG distribution for selected values of the parameters. $\theta, \lambda, \alpha\ and\ \beta\ with\ \gamma = 1$.

CDF of the TLG distribution for selected values of the parameters.

The graphical representation of the cumulative function for different possible values of the parameter is shown in figure. 4.2, which is always an increasing function.

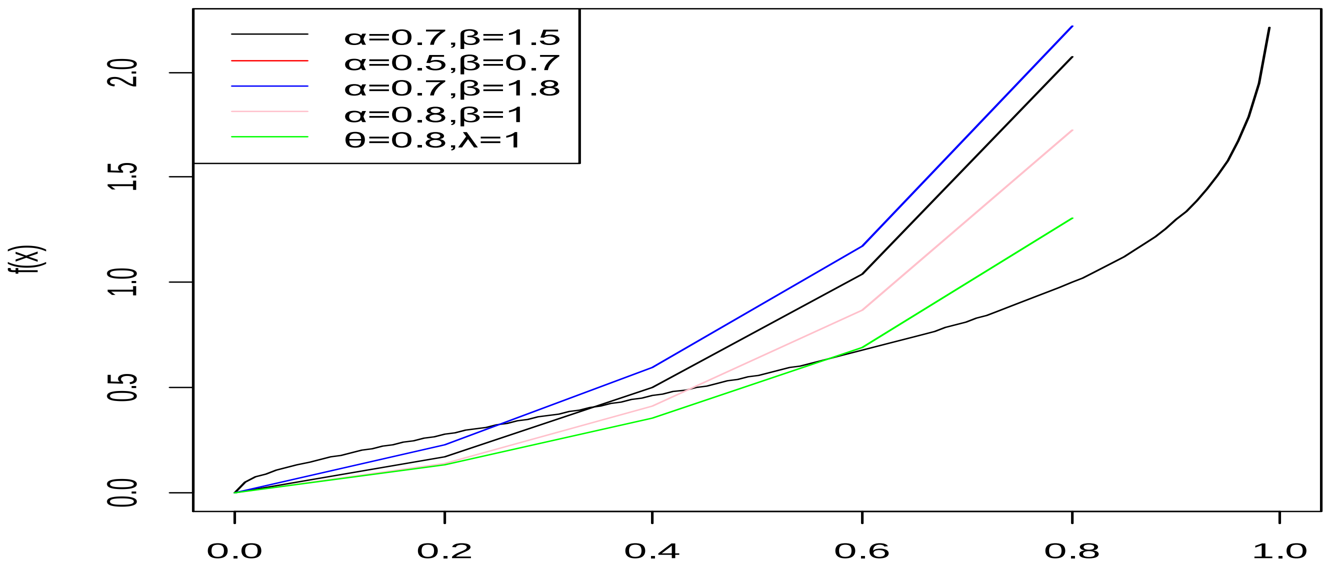

Fig. 4.3: Quantile Function of TLG Distribution

The quantile function for selected parameter values of the θ, λ, α *and* β *with* $\gamma = 1$.

The plot for the quantile function reveals that the quantile function is positively skewed and therefore will be a good model for a positively skewed dataset.

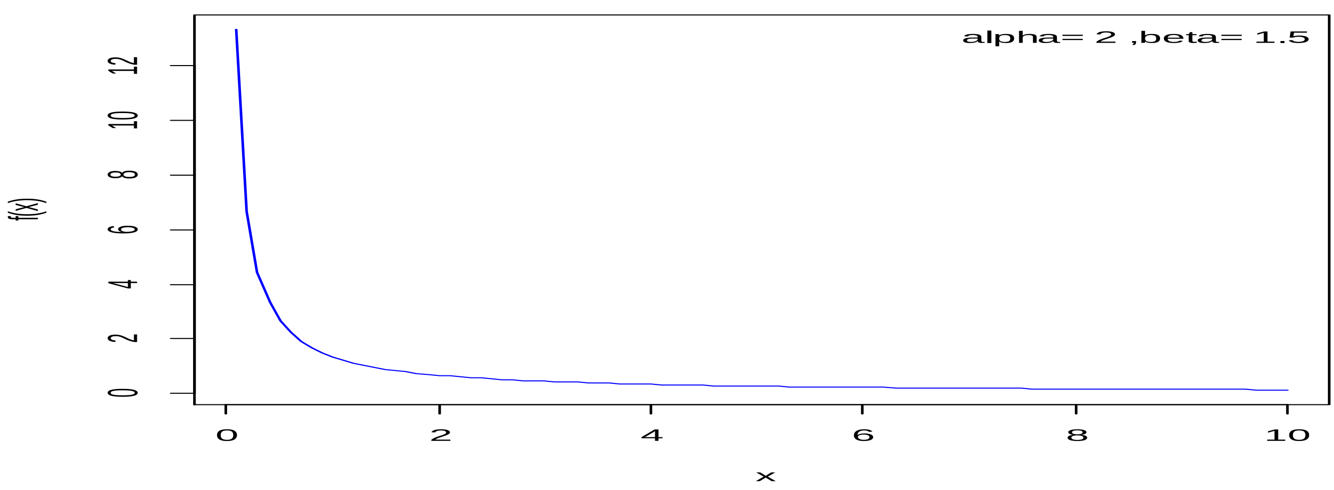

Fig. 4.4: Hazard plot of the TLG Distribution

Figure 4.4 illustrates some of the possible shapes of the hazard function of the TLG distribution for selected values of the parameters. α *and* β *with* $\gamma = 1$.

Hazard functions of the TLG distribution for selected values of the parameters.

The graphical representation of the hazard function for different possible values of the parameter is shown in figure 4.4, which is always a decreasing function.

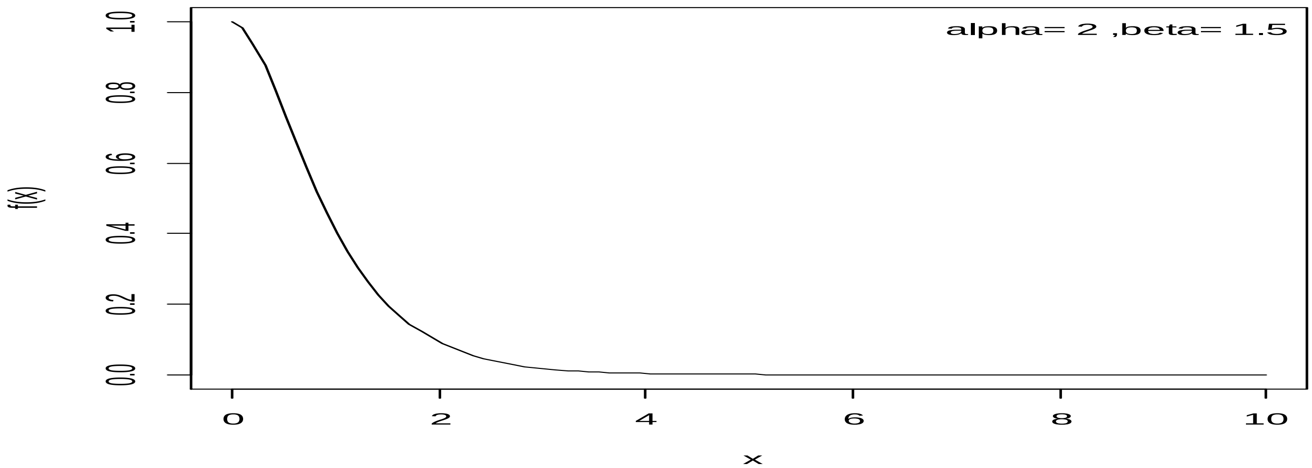

Fig. 4.5: The Survival plot of the TLG Distribution

Figure 4.5 illustrates some of the possible shapes of the survival function of the TLG distribution for selected values of the parameters. α and β $with$ $\gamma = 1$.

Survival function of the TLG distribution for selected values of the parameters.

The graphical representation of the survival function for different possible values of the parameter is shown in figure 4.5, which is always an increasing function.

Model selection was perform using the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), and Hannan-Quinn Information Criterion (HQIC). The model with the minimum value of AIC or BIC, CAIC, and HQIC was selected as the best model to fit the data.

Table 4.2.3: Goodness of Fits Statistics for the Datasets

Data sets	**Models**	**MLEs**				
		AIC	BIC	$CAIC$	$HQIC$	Value
Sum of skin fold data	TLGD ECTED WGED MOLED	1649.084 1966.286 2066.381 1942.726	1662.317 1979.519 2076.306 1952.651	1649.287 1966.489 2066.503 1942.847	1654.438 1971.64 2070.397 1946.742	820.542 979.1428 1030.191 968.363
Duncan data	TLGD ECTED WGED MOLED	800.146 866.945 852.3523 855.1786	810.5666 877.3657 860.1678 862.9941	800.567 867,3661 852.6023 855.4286	804.3634 871.1624 855.5154 858.3416	396.073 429.4725 423.1761 424.5893

10 Conclusions

We used the adequacy model package in R-Console, and goodness-of-fit analytical measures were used to compare the performances of the models. From the table 4.5, the lowest value of the mentioned statistics corresponds to the Transmuted Lomax Gamma distribution (TLGD), which means that it fits the datasets better than the other three models. Therefore, it should be used for fitting the subject data rather than the other three methods.

11 Conclusion

In this study, we introduce a new class of distributions called the Transmuted Lomax -Gamma (TLG) family of distributions. This family can extend several widely known models. For instance, we considered the gamma distribution as the baseline distribution. We investigated some of the structural properties of the gamma distribution for the density function using integration expansion. Some of the derived properties include the moment -generating function, Reni and entropies, quantile function, and order statistics. The parameters were estimated using the maximum likelihood estimation method. The parameter estimates and the associated analytical measures showed that the new model based on the two datasets found that our developed model performs better in fitting the dataset than all the other distributions that we compared it with. Therefore, we conclude that based on the datasets used in this study, our model should be chosen.

In additionr, our distribution has a PDF that can be right (positively) skewed, and it has a very heavy fat tail; thus, it can capture many types and features of a dataset. The hazard rate function can also increase (shape), or decreasing (shape).

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